BUBBLING ALONG BOUNDARY GEODESICS FOR LANE-EMDEN-FOWLER PROBLEM NEAR THE SECOND CRITICAL EXPONENT IN DIMENSIONS 6 AND 7

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ABSTRACT. We construct geodesics bubbling solutions along a nondegenerate closed geodesic $\Gamma \subset \partial\Omega$ with negative inner normal curvature for the Lane-Emden-Fowler problem

$$\begin{cases} \Delta u + u^{\frac{n+1}{n-3}-\epsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$ with $n = 6, 7$, $\partial\Omega$ is smooth and bounded, and $\epsilon > 0$ is a small parameter. We prove that there exists a solution u_ϵ such that $|\nabla u_\epsilon|^2$ converges to the Dirac measure on Γ as $\epsilon \to 0^+$ with ϵ satisfying a certain non-resonance condition. Hence we extend the result in [del Pino, Musso and Pacard, J. Eur. Math. Soc. 12 (2010), 1553-1605] to lower dimension case $n = 6, 7$. The new ingredient in our approach is a new inner-outer gluing method which works for all dimensions $n \geq 6$.

1. Introduction

1.1. **The problem.** In this paper, we consider the classical Lane-Emden-Fowler problem (see [23])

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p > 1$ and $\Omega \subset \mathbb{R}^n$ with smooth bounded boundary.

For $1 < p < \frac{n+2}{n-2}$, compactness of Sobolev embedding yields the existence of solution by minimizing the following functional

$$S(p) = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2}{\left(\int_\Omega |u|^{p+1}\right)^{\frac{2}{p+1}}}.$$

For subcritical case, an interesting phenomenon is the point bubbling if $p = \frac{n+2}{n-2} - \epsilon$ as $\epsilon \to 0^+$. To be more precise, for $\epsilon > 0$ small, there exists a solution to (1.1) in the form

$$u_\epsilon(x) = \mu_\epsilon^{-\frac{n-2}{2}} w_n\left(\frac{x - x_\epsilon}{\mu_\epsilon}\right) + o(1), \quad \mu_\epsilon \sim \epsilon^{\frac{1}{n-2}},$$

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as $\epsilon \to 0^+$. Here w_n is the Aubin-Talenti bubble

$$w_n(x) = \left(\frac{c_n}{1+|x|^2}\right)^{\frac{n-2}{2}} \tag{1.2}$$

which is the bounded radial solution of

$$\Delta w + w^{\frac{n+2}{n-2}} = 0, \quad \text{in } \mathbb{R}^n, \tag{1.3}$$

where $c_n = \sqrt{n(n-2)}$ (see [1,35]). The blow-up point x_ϵ concentrates on a nondegenerate critical point x_0 of Robin's function of Ω. For more related results see for example [3,7,13,17,22,24,25,33] and the references therein.

For $p \geq \frac{n+2}{n-2}$, Pohozaev identity [32] implies that there is no solution of (1.1) if the domain Ω is star-shaped. For Ω with other geometry structures, solutions may exist. For example, in [26], Kazdan and Warner proved that if Ω is a symmetric annulus, then the compactness of Sobolev embedding can be recovered for all $p > 1$ within the radial function space, which verifies the existence of solutions to (1.1). In [2], Bahri and Coron obtained the existence of solutions to (1.1) for $p = \frac{n+2}{n-2}$ if Ω has nontrivial topology. On the other hand, in [6], Brezis and Nirenberg recovered the compactness by suitable linear perturbations for the critical exponent $p = \frac{n+2}{n-2}$. For $p > \frac{n+2}{n-2}$, variational method seems difficult to show the existence. A question raised by Rabinowitz, stated by Brezis in [5], is whether the nontrivial topology of the domain is sufficient for the solvability of (1.1) for $p > \frac{n+2}{n-2}$. However, Passaseo [31] constructed a counterexample for this question by choosing the domain Ω to be a thin tubular neighborhood of a copy of the unit sphere $\mathbb{S}^{n-2}$ in $\mathbb{R}^n$ $(n \geq 4)$ with $p \geq \frac{n+1}{n-3}$. Here $\frac{n+1}{n-3}$ is called the *second critical exponent*, which is strictly larger than the critical exponent $\frac{n+2}{n-2}$.

In an interesting paper [18], del Pino, Musso and Pacard first constructed solutions to (1.1) when $p = \frac{n+1}{n-3} - \epsilon$ with $\epsilon > 0$ sufficiently small and $n \geq 8$. More precisely, they proved that if $\partial\Omega$ contains a nondegenerate closed geodesic Γ with strictly negative inner normal curvature, then there exists a solution of (1.1) with a concentration behavior as $p \to (\frac{n+1}{n-3})^-$ in the form of bubbling line which collapses to Γ. A typical example of such domain is that Ω has a convex hole. This phenomenon is called *line bubbling*. Note that the argument in [18] relies crucially on the dimension restriction $n \geq 8$. The line bubbling phenomenon has also been discovered in supercritical problems with $p = \frac{n+1}{n-3} \pm \epsilon$ on compact Riemannian manifold without boundary [12]. The concentration at higher dimensional boundary submanifolds was investigated in [21]. Also, the constructions in [12] and [21] rely on a similar technical restriction that the codimension of the concentration submanifold is no less than 7.

We remark that point bubbling is determined by the Green's function which relies on the global information of the domain Ω, whereas from the construction of [18], line bubbling only depends on the local structure of the domain near the concentration curve Γ.

In this paper, we shall investigate the line bubbling for lower dimension case $n = 6,\ 7$. More precisely, we study the following problem

$$\begin{cases} \Delta u + u^{\frac{n+1}{n-3}-\epsilon} = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

Let $\Gamma \subset \partial\Omega \subset \mathbb{R}^n$ be a closed nondegenerate (see Definition 1.1 below) geodesic with global negative curvature. We assume that a non-resonance condition

$$|k^2 \epsilon^{\frac{2(n-2)}{n-3}} - \kappa^2| > \delta \epsilon^{\frac{n-2}{n-3}}, \quad \forall k \in \mathbb{Z}_+ \tag{1.5}$$

holds, where $\delta > 0$ is a constant, and $\kappa > 0$ is a constant depending only on Γ (see (7.25)). Note that the resonance phenomenon has been found in higher dimensional concentration for many elliptic problems, see for example [29], [30], [28], [27], [14], [18], [12] and the references therein.

For simplicity, in the rest of this paper, we denote

$$N = n - 1 \text{ and } p = \frac{n+1}{n-3} = \frac{N+2}{N-2}.$$

1.2. **Main result.** To describe our main result precisely, we introduce some geometry notations. Let $\partial\Omega$ endow with the metric induced by the Euclidean metric and $\bar{\nabla}$ be the associated connection. Near the geodesic Γ, we introduce the following *Fermi coordinates*. Let $q \in \partial\Omega$ and we split

$$T_q\partial\Omega = T_q\Gamma \oplus N_q\Gamma$$

into the tangent and normal bundles over Γ. Assume that Γ is parameterized by arclength x_0 with $x_0 \to \gamma(x_0)$, and the length of Γ is $2l$. Let E_0 be a unit tangent vector to Γ, E_i, $i = 1, \cdots, N-1$, be an orthonormal basis of $N_q\Gamma$ which are parallel along Γ, namely

$$\bar{\nabla}_{E_0} E_i = 0, \quad i = 1, \cdots, N-1.$$

Since Γ is geodesic, it holds that

$$\bar{\nabla}_{E_0} E_0 = 0.$$

In a neighborhood of Γ in $\partial\Omega$, using exponential map $\exp^{\partial\Omega}$ on $\partial\Omega$, we introduce a local coordinates

$$F(x_0, \bar{x}) := \exp^{\partial\Omega}_{\gamma(x_0)}(x_i E_i), \quad \bar{x} := (x_1, x_2, \cdots, x_{N-1}),$$

where we use Einstein summation over $i = 1, \cdots, N-1$ for simplicity. In the neighborhood of Γ in $\bar{\Omega}$, we give a local coordinates

$$G(x_0, x) := F(x_0, \bar{x}) - x_N \mathbf{n}(F(x_0, \bar{x})), \quad x = (\bar{x}, x_N) \in \mathbb{R}^N,$$

where x is in a small neighborhood of 0 and $\mathbf{n}$ is the outward unit normal. Assume that the curvature of Γ is given by

$$\partial^2_{x_0}\gamma = \bar{h}_{00}\mathbf{n},$$

where $\bar{h}_{00}$ is a strictly positive function on Γ.

Definition 1.1. *In local coordinates, we say that the geodesic Γ is nondegenerate if*

$$-\bar{d}_k'' + \sum_{j=1}^{N-1} \langle \bar{R}(E_0, E_j)E_0, E_k\rangle \bar{d}_j = 0, \quad x_0 \in [-l, l], \ k = 1, \cdots, N-1, \tag{1.6}$$

has only the trivial $2l$-periodic solution $\bar{d} \equiv 0$, where $\bar{R}$ denotes the Ricci tensor on $\partial\Omega$.

Our main result is as follows.

Theorem 1.1. *Let $n = 6, 7$ and $\Omega \subset \mathbb{R}^n$ be a domain with smooth bounded boundary $\partial\Omega$. Assume $\Gamma \subset \partial\Omega$ is a closed nondegenerate geodesic with negative inner normal curvature. Then for all $\epsilon > 0$ sufficiently small satisfying the non-resonance condition (1.5) with $\delta > 0$ fixed, there exists a solution u_ϵ of (1.4) such that*

$$|\nabla u_\epsilon|^2 \rightharpoonup S_{n-1}^{\frac{n-1}{2}} \delta_\Gamma, \quad \text{as } \epsilon \to 0^+,$$

in the measure sense, where δ_Γ denotes the Dirac measure supported on Γ. Furthermore, u_ϵ is in the following form

$$u_\epsilon(x_0, x) = \mu_\epsilon^{-\frac{N-2}{2}} w\left(\frac{x - d_\epsilon}{\mu_\epsilon}\right) + o(1),$$

where μ_ϵ and d_ϵ are defined in (3.2)-(3.4), $w := w_N$ is the standard bubble given in (1.2) and $N = n - 1$.

1.3. **Main idea of proof.** Our proof is based on the so-called **inner-outer gluing procedure** which is a very useful tool in constructing higher dimensional concentrating solutions for various elliptic problems, see, for example, [14–16, 18] and the references therein. Recently, this method has been successfully applied to various critical heat equations, see, for example, [8–11, 19, 20, 34] and the references therein. One of the key ingredients in this method is to prove proper a priori estimates for associated linearized operators. Inspired by the linear theory in [18] and [11, Section 4], we develop a linear theory in the elliptic setting which shares the similar flavor of parabolic problems. Formally, in our problem, the tangential direction y_0 plays a similar role as the time variable. More precisely, we consider the projected equation

$$\begin{cases} a_0\partial_0^2\phi + \Delta_y\phi + \tilde{\mathcal{A}}\phi = h + \sum_{j=0}^{N+1} c_j(\rho y_0) Z_j(y), \text{ in } \mathbb{S}_\rho \times D_R, \\ \phi(y_0, y) = 0, \ \forall (y_0, y) \in \partial(\mathbb{S}_\rho \times D_R), \end{cases} \tag{1.7}$$

where a_0 is a positive smooth $2l$-periodic function of ρy_0, $\tilde{\mathcal{A}}$ is a small coefficient operator given by Lemma 3.1 below, $\rho = \epsilon^{\frac{N-1}{N-2}}$, $\mathbb{S}_\rho$ is the circle parameterized by $y_0 \in [-\frac{l}{\rho}, \frac{l}{\rho}]$,

$$D_R = \{y = (\bar{y}, y_N) \in \mathbb{R}^N : |\bar{y}| < 2R, -\frac{d_{\epsilon,N}}{\mu_\epsilon}(\rho y_0) < y_N < 2R\}$$

with $R = R(\epsilon) = \epsilon^{-\theta_*}$ and θ_* as in (7.13), Z_0 is the first eigenfunction defined in (3.18), and Z_i ($i = 1, \cdots, N, N+1$) are the bounded kernel functions of the linearized operator of equation (1.3) around the standard bubble $w = w_N$ defined in (1.2), namely,

$$Z_j = \partial_j w \text{ for } j = 1, \cdots, N, \text{ and } Z_{N+1} = \frac{N-2}{2} w + x \cdot \nabla w,$$

(see for example [4]). Define the L^∞-weighted norms

$$\begin{aligned} \|\phi\|_\sigma &:= \sup_{\mathbb{S}_\rho \times D_R} \langle y \rangle^\sigma |\phi(y_0, y)| + \sup_{\mathbb{S}_\rho \times D_R} \langle y \rangle^{1+\sigma} |\nabla_y \phi(y_0, y)|, \\ \|h\|_{2+\sigma} &:= \sup_{\mathbb{S}_\rho \times D_R} \langle y \rangle^{2+\sigma} |h(y_0, y)|, \end{aligned} \tag{1.8}$$

where $0 < \sigma \le N-4$ with $N = n-1$ and $\langle y\rangle := \sqrt{1+|y|^2}$. It will be shown in Section 5 (see Proposition 5.2 below) that there exists a solution ϕ to equation (1.7) satisfying the following estimate

$$|\phi| \lesssim R^{\tau-\sigma}\langle y\rangle^{-\tau}\|h\|_{2+\sigma} \tag{1.9}$$

where $2 < \tau < N-2$. The main difference between Proposition 5.2 and the linear theory in [18] is as follows: to obtain similar a prior estimates as in [18], the authors assume that the solution ϕ is orthogonal to Z_i $(i = 0, 1, \cdots, N, N+1)$ in which a dimension restriction $n \ge 8$ is needed to guarantee the integrability of orthogonality conditions in $\mathbb{R}^N$, whereas Proposition 5.2 is established by first proving a fast decaying version (see Proposition 5.1) of the linear equation (1.7), then we apply it to the slow decaying version (see Proposition 5.2) and get the desired estimate (1.9). As a consequence, in the intermediate region $|y| \sim R$, estimate (1.9) implies

$$\|\phi\|_\sigma \lesssim \|h\|_{2+\sigma},$$

while in the interior, the estimate we get for the solution ϕ is deteriorated for low dimension case $n = 6, 7$, namely that $R^{\tau-\sigma}$ appears in front. However, by choosing R appropriately and some further efforts, the linear theory is sufficient for us to carry out the inner-outer gluing scheme.

We note that our new gluing method can also be used to reduce the co-dimensions in [12] and [21]. We remark that, for dimension $n = 4, 5$ (i.e. $N = 3, 4$), similar result as Proposition 5.2 does not hold and we need to make essential changes in the argument. Furthermore in these low dimensions the equation for the scaling parameter may become nonlocal. We will return to this topic in a future work.

This paper is organized as follows. In Section 2, we recall some basic geometry notations for local coordinates near the geodesic Γ. Section 3 is devoted to constructing the approximate solution and computing the size of the error. In Section 4, we set up the inner-outer gluing scheme and solve the outer problem. Before we solve the reduced projected inner problem in Section 6, we shall develop a linear theory of the associated linear problem in Section 5. Finally in Section 7, we adjust the parameter functions such that the reduced system $c_j(\rho y_0) = 0$ in (1.7) is satisfied for all y_0 and $j = 0, 1, \cdots, N+1$ and prove Theorem 1.1.

Throughout this paper, the notation "$\lesssim$" always denotes "$\le c$" where the constant $c > 0$ may differ from line to line but it is independent of ϵ.

2. Geometric Settings

In this section, we recall some geometric notations and results for the problem as in [18]. We refer the readers to [18] for detailed computations.

Consider the metric $\bar{g}$ on $\partial\Omega$ induced by the Euclidean metric in $\mathbb{R}^n$. Denote the associated connection by $\bar{\nabla}$. Then we introduce the *Fermi coordinates* in a neighborhood of Γ on $\partial\Omega$. Given $q \in \Gamma$, we have the natural splitting

$$T_q\partial\Omega = T_q\Gamma \oplus N_q\Gamma$$

into the tangent and normal bundles over Γ. Assume that Γ is parameterized by the arclength x_0 with $x_0 \mapsto \gamma(x_0)$. Denote E_0 by the unit tangent vector to Γ. In a neighborhood of $q \in \Gamma$, we denote $E_1, \cdots, E_{N-1}$ by an orthonormal basis of $N_q\Gamma$. We can assume that for each $i = 1, \cdots, N-1$

$$\bar{\nabla}_{E_0}E_i = 0.$$

Since Γ is a geodesic,

$$\bar{\nabla}_{E_0} E_0 = 0.$$

Define

$$F(x_0, \bar{x}) = \exp^{\partial\Omega}_{\gamma(z_0)}(x_i E_i), \ \bar{x} = (x_1, \cdots, x_{N-1}),$$

where $\exp^{\partial\Omega}$ is the exponential map on $\partial\Omega$ and the summation is from 1 to $N-1$.

The above Fermi coordinates are defined such that $\bar{g}_{ab} = \delta_{ab}$ along Γ, where $\bar{g}_{ab}$ is the coefficient of $\bar{g}$. Then the higher order terms in the Taylor expansion of the metric coefficients are estimated as follows, whose proof can be found in [18].

Proposition 2.1. *At $q = F(x_0, \bar{x})$, the following estimates hold*

$$\begin{aligned}
\bar{g}_{00} &= 1 + \langle \bar{R}(E_0, E_k)E_0, E_l \rangle x_k x_l + \mathcal{O}(|\bar{x}|^3),\\
\bar{g}_{0i} &= \mathcal{O}(|\bar{x}|^2),\\
\bar{g}_{ij} &= \delta_{ij} + \frac{1}{3}\langle \bar{R}(E_i, E_k)E_j, E_l \rangle x_k x_l + \mathcal{O}(|\bar{x}|^3),
\end{aligned}$$

where $i, j, k, l = 1, \cdots, N-1$, $\bar{R}$ is the curvature tensor and $\mathcal{O}(|\bar{x}|^s)$ is a smooth function not involving any term up to order s in x_i, $i = 1, \cdots, N-1$.

For simplicity, we denote

$$R_{ijkl} = \langle \bar{R}(E_i, E_j)E_k, E_l \rangle. \tag{2.1}$$

In order to parameterize a neighborhood near Γ, we define the following coordinates $(x_0, x) \in \mathbb{R}^{N+1}$

$$G(x_0, x) = F(x_0, \bar{x}) - x_N \mathbf{n}(F(x_0, \bar{x})), \ x = (\bar{x}, x_N) \in \mathbb{R}^N,$$

with x close to 0 and $\mathbf{n}$ denotes the unit outward normal.

The coefficients of the Euclidean metric in these coordinates

$$g_{aN} = g_{Na} = 0, \ g_{NN} = 1 \quad \text{for } a = 0, \cdots, N-1. \tag{2.2}$$

Moreover, it holds that

$$g_{ab} = \bar{g}_{ab} + 2\bar{h}_{ab} x_N + \bar{k}_{ab} x_N^2 + O(x_N^3) \quad \text{for } a, b = 0, \cdots, N-1,$$

where $\bar{g}$ is the metric on $\partial\Omega$, $\bar{h}$ is the second fundamental form of $\partial\Omega$ with

$$\bar{h}_{ab} = -E_b \cdot \nabla_{E_a} \mathbf{n} = -E_a \cdot \nabla_{E_b} \mathbf{n}, \tag{2.3}$$

and

$$\bar{k}_{ab} = (\bar{h} \otimes \bar{h})_{ab} = \sum_{c,d} \bar{h}_{ac} \bar{g}^{cd} \bar{h}_{db}. \tag{2.4}$$

We remark that the normal curvature along the geodesic Γ in this setting is

$$\partial^2_{x_0} \gamma = \nabla_{E_0} E_0 = \bar{h}_{00} \mathbf{n}.$$

Based on the above settings, we now compute the expansion of the Laplace-Beltrami operator

$$\Delta = \frac{1}{\sqrt{|g|}} \partial_{x_\alpha}(\sqrt{|g|} g^{\alpha\beta} \partial_{x_\beta}) = g^{pq} \partial_{x_\alpha} \partial_{x_\beta} + \partial_p g^{\alpha\beta} \partial_{x_\beta} + \frac{1}{2} \mathrm{Tr}_g(\partial_{x_\alpha} g) g^{\alpha\beta} \partial_{x_\beta}$$

with the summation taken over $\alpha, \beta = 0, \cdots, N$. By (2.2), we have

$$\Delta = \partial^2_{x_N} + \frac{1}{2} \mathrm{Tr}_g(\partial_{x_N} g) \partial_{x_N} + g^{ab} \partial_{x_a} \partial_{x_b} + \partial_{x_a} g^{ab} \partial_{x_b} + \frac{1}{2} \mathrm{Tr}_g(\partial_{x_a} g) g^{ab} \partial_{x_b},$$

where the summation is taken over $a, b = 0, \cdots, N-1$.

Direct computations give the decomposition as follows

$$
\begin{aligned}
\Delta = {} & \partial_{x_0}^2 + \sum_j \partial_{x_j}^2 + \partial_{x_N}^2 + A^{00}\partial_{x_0}^2 + \sum_j A^{0j}\partial_{x_0}\partial_{x_j} \\
& + \sum_{i,j}\left[-\frac{1}{3}\sum_{k,l}\langle \bar{R}(E_i,E_k)E_j,E_l\rangle x_k x_l - 2\bar{h}_{ij}x_N + A^{ij}\right]\partial_{x_i}\partial_{x_j} \\
& + B^0\partial_{x_0} + \sum_j\left[\sum_k\left(\frac{2}{3}\langle \bar{R}(E_i,E_j)E_i,E_k\rangle + \langle \bar{R}(E_0,E_j)E_0,E_k\rangle\right)x_k + B^j\right]\partial_{x_j} \\
& + (\mathrm{Tr}_{\bar{g}}\bar{h} - \mathrm{Tr}_{\bar{g}}\bar{k}x_N + B^N)\partial_{x_N},
\end{aligned}
$$

where $\bar{R}$, $\bar{g}$, $\bar{h}$ and $\bar{k}$ only depend on x_0. The metric $\bar{g}$, tensors $\bar{h}$ and $\bar{k}$ only depend on x_0. All the rest functions A^{ij} and B^j depend on $x_0, x_1, \cdots, x_N$ and have further decompositions as in [18, (4.13)]. For the reader's convenience, we list below

$$
\begin{aligned}
A^{00} &= A_N^{00}x_N + \sum_{k,l} A_{kl}^{00}x_k x_l, \\
A^{ij} &= A_N^{ij}x_N^2 + \left(\sum_k A_{Nk}^{ij}x_k\right)x_N + \sum_{k,l,m} A_{kl}^{ij}x_k x_l x_m, \\
A^{0j} &= A_N^{0j}x_N + \sum_{k,l} A_{kl}^{0j}x_k x_l, \\
B^0 &= B_N^0 x_N + \sum_k B_k^0 x_k, \\
B^j &= B_N^j x_N + \sum_{k,l} B_{kl}^j x_k x_l, \\
B^N &= B_N^N x_N^2 + \left(\sum_k B_k^N x_k\right)x_N + \sum_j B_j^N x_j,
\end{aligned}
\tag{2.5}
$$

where all the functions in (2.5) are smooth and depend on $x_0, \cdots, x_N$.

3. Construction of the approximate solution

In this section, we shall construct an approximate solution to the following problem

$$
\begin{cases}
\Delta u + u^{\frac{N+2}{N-2}-\epsilon} = 0, & \text{in } \Omega \\
u = 0, & \text{on } \partial\Omega.
\end{cases}
$$

Our first approximate solution is based on the Aubin-Talenti bubble w satisfying

$$
\Delta w + w^{\frac{N+2}{N-2}} = 0 \text{ in } \mathbb{R}^N,
$$

namely,

$$
w(x) = \left(\frac{c_N}{1+|x|^2}\right)^{\frac{N-2}{2}}
$$

with $c_N = \sqrt{N(N-2)}$.

3.1. **Scaling coordinates near the geodesic.** We shall rescale and translate the bubble along a curve close to the geodesic Γ. Let $(x_0, x) \in \mathbb{R}^{N+1}$ be the local coordinates near the geodesic Γ. We perform the following change of variables

$$(y_0, y) := \left(\frac{x_0}{\rho}, \frac{x - d_\epsilon}{\mu_\epsilon}\right)$$

and

$$u(G(x_0, x)) = \mu_\epsilon^{-\frac{N-2}{2}} v\left(\frac{x_0}{\rho}, \frac{x - d_\epsilon}{\mu_\epsilon}\right), \tag{3.1}$$

where $v = v(y_0, y)$, $\rho = \epsilon^{\frac{N-1}{N-2}}$,

$$\mu_\epsilon(x_0) = \rho\tilde{\mu}_\epsilon(x_0), \ d_\epsilon(x_0) = \epsilon\tilde{d}_\epsilon(x_0) \tag{3.2}$$

with

$$\tilde{\mu}_\epsilon(x_0) = \mu_\epsilon^0(x_0) + \epsilon\mu(x_0), \ \tilde{d}_{\epsilon,j}(x_0) = \epsilon d_j(x_0), \text{ for } j = 1, \cdots, N-1, \tag{3.3}$$

and $\tilde{d}_{\epsilon,N}(x_0) = d_{\epsilon,N}^0(x_0) + \epsilon d_N(x_0)$. In the above definitions, $\mu_\epsilon^0(x_0)$ and $d_{\epsilon,N}^0(x_0)$ are given by

$$\mu_\epsilon^0(x_0) = \mu_0(x_0) + \epsilon^{\frac{1}{N-2}}\mu_1(x_0), \ d_{\epsilon,N}^0(x_0) = d_{0,N}(x_0) + \epsilon^{\frac{1}{N-2}} d_{1,N}(x_0) \tag{3.4}$$

with $\mu_0(x_0) = \frac{\alpha}{\bar{h}_{00}(x_0)}$, $d_{0,N}(x_0) = \frac{\beta}{\bar{h}_{00}(x_0)}$, where positive constants α and β only depend on N, and $\bar{h}_{00}$ is the normal curvature along Γ. We assume Γ has globally negative curvature, namely,

$$\partial_{x_0}^2 \gamma = \bar{h}_{00}\mathbf{n},$$

where $\bar{h}_{00}$ is a smooth and strictly positive function along Γ, and $\mathbf{n}$ is the outward unit normal.

The norms of the parameter functions $\mu(x_0)$ and $d(x_0) = (d_1, \cdots, d_N)$ in $(-l, l)$ are defined as follows

$$\|\mu\|_a = \|\epsilon^{\frac{N}{N-2}}\mu''\|_\infty + \|\epsilon^{\frac{N}{2(N-2)}}\mu'\|_\infty + \|\mu\|_\infty, \tag{3.5}$$

and

$$\|d\|_d = \|d_N\|_b + \sum_{j=1}^{N-1} \|d_j\|_c, \tag{3.6}$$

with

$$\|d_N\|_b = \|\epsilon d_N''\|_\infty + \|\epsilon^{\frac{1}{2}} d_N'\|_\infty + \|d_N\|_\infty \tag{3.7}$$

and

$$\|d_j\|_c = \|d_j''\|_\infty + \|d_j'\|_\infty + \|d_j\|_\infty, \text{ for } j = 1, \cdots, N-1. \tag{3.8}$$

In the above definitions, the prime denotes $\frac{d}{dx_0}$.

After the change of variables (3.1), the original cylinder close to Γ is transformed into the following region

$$(y_0, y) \in \mathcal{D} := \left\{(y_0, \bar{y}, y_N) : -\frac{d_{\epsilon,N}}{\mu_\epsilon} < y_N < \frac{\hat{\delta}}{\rho}, |\bar{y}| < \frac{\hat{\delta}}{\rho}\right\} \tag{3.9}$$

with some fixed $\hat{\delta} > 0$.

3.2. **Equation in the local scaling coordinates.** After performing the change of variable (3.1), the Laplacian becomes

$$\mu_\epsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v),$$

where $\mathcal{A}v = a_0 \partial_0^2 v + \Delta_y v + \tilde{\mathcal{A}} v$ with $a_0 = \tilde{\mu}_\epsilon^2$ and $\tilde{\mu}_\epsilon$ is defined in (3.2). The differential operator $\tilde{\mathcal{A}}$ can be expressed as

$$\tilde{\mathcal{A}} v = \sum_{\alpha,\beta} a_{\alpha,\beta} \partial_{\alpha,\beta} v + \sum_\alpha b_\alpha \partial_\alpha v + cv$$

with

$$a_{\alpha,\beta} = O(\epsilon + \rho^2 |y|^2), \text{ if } \alpha \neq 0, \beta \neq 0, \; a_{0,\beta} = O(\epsilon), \; a_{0,0} = 0$$

and

$$b_\alpha = \rho O(\epsilon + \rho |y|), \; c = \rho^2 O(1).$$

The more specific expression of $\mathcal{A}$ is given by the following Lemma, whose proof can be found in [18, Lemma 5.1].

Lemma 3.1. *After the change of variables (3.1), it holds that*

$$\mu_\epsilon^{\frac{N+2}{2}} \Delta u = \mathcal{A}(v) := a_0 \partial_0^2 v + \Delta_y v + \sum_{k=0}^{5} \mathcal{A}_k v + B(v),$$

where

$$\begin{aligned} \mathcal{A}_0(v) = & (\mu_\epsilon')^2 \left[D_{yy} v[y]^2 + 2(1+\gamma) D_y v[y] + \gamma(1+\gamma) v \right] \\ & + \mu_\epsilon' \left[D_{yy} v[y] + \gamma D_y v \right] [d_\epsilon'] + D_{yy} v [d_\epsilon']^2 \\ & - 2\mu_\epsilon \left[\epsilon^{-\frac{N-1}{N-2}} D_y(\partial_0 v)[\mu_\epsilon' y + d_\epsilon'] + \gamma \mu_\epsilon' \epsilon^{-\frac{N-1}{N-2}} \partial_0 v \right] \\ & - \mu_\epsilon D_y v [d_\epsilon''] - \mu_\epsilon \mu_\epsilon'' (\gamma v + D_y v[y]), \end{aligned} \tag{3.10}$$

$$\begin{aligned} \mathcal{A}_1 v = \sum_{i,j} \Big[& -\frac{1}{3} R_{ikjl} (\mu_\epsilon y_k + d_{\epsilon,k})(\mu_\epsilon y_l + d_{\epsilon,l}) - 2\bar{h}_{ij} (\mu_\epsilon y_N + d_{\epsilon,N}) \\ & + \sum_k a_{Nk}^{ij} (\mu_\epsilon y_k + d_{\epsilon,k})(\mu_\epsilon y_N + d_{\epsilon,N}) \Big] \partial_{ij} v, \end{aligned} \tag{3.11}$$

$$\begin{aligned} \mathcal{A}_2 v = \sum_j \Big[& -4\bar{h}_{0j} (\mu_\epsilon y_N + d_{\epsilon,N}) \\ & \times (-D_y(\partial_j v)[d] + \mu_\epsilon \epsilon^{-\frac{N-1}{N-2}} \partial_{0j} v - (\gamma \partial_j v + D_y(\partial_j v)[y]) \mu_\epsilon') \Big], \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mathcal{A}_3 v = & \left(\sum_k b_k^0 [\mu_\epsilon y_k + d_{\epsilon,k}] + b_N^0 (\mu_\epsilon y_N + d_{\epsilon,N}) \right) \\ & \times \left\{ \mu_\epsilon \left[-D_y v [d_\epsilon'] + \mu_\epsilon \epsilon^{-\frac{N-1}{N-2}} \partial_0 v - \mu_\epsilon' (\gamma v + D_y v[y]) \right] \right\}, \end{aligned} \tag{3.13}$$

$$\mathcal{A}_4 v = \sum_j \left[\sum_k (\frac{2}{3} R_{ijik} + R_{0j0k})(\mu_\epsilon y_k + d_{\epsilon,k}) + b_N^j (\mu_\epsilon y_N + d_{\epsilon,N}) \right] \mu_\epsilon \partial_j v, \tag{3.14}$$

$$\mathcal{A}_5 v = \left(\mathrm{Tr}_{\bar{g}} \bar{h} - \mathrm{Tr}_{\bar{g}} \bar{k} (\mu_\epsilon y_N + d_{\epsilon,N}) \right) \mu_\epsilon \partial_N v, \tag{3.15}$$

$$
\begin{aligned}
B(v) =& O\left(|\mu_\epsilon \bar{y} + \bar{d}_\epsilon|^2 + (\mu_\epsilon y_N + d_{\epsilon,N}) + (\mu_\epsilon y_N + d_{\epsilon,N})(\mu_\epsilon \bar{y} + \bar{d}_\epsilon)\right) \mathcal{A}_0(v) \\
&+ O\left(|\mu_\epsilon \bar{y} + \bar{d}_\epsilon|^3 + (\mu_\epsilon y_N + d_{\epsilon,N})|\mu_\epsilon \bar{y} + \bar{d}_\epsilon|^2 + (\mu_\epsilon y_N + d_{\epsilon,N})^2\right) \partial_{ij} v \\
&+ O\left(|\mu_\epsilon \bar{y} + \bar{d}_\epsilon|^2 + (\mu_\epsilon y_N + d_{\epsilon,N})|\mu_\epsilon \bar{y} + \bar{d}_\epsilon| + (\mu_\epsilon y_N + d_{\epsilon,N})^2\right) \\
&\times \Big[\mu_\epsilon \epsilon^{-\frac{N-1}{N-2}} \partial_{0j} v + \mu_\epsilon \epsilon^{-\frac{N-1}{N-2}} \partial_0 v - D_y(\partial_j v)[d_\epsilon] \\
&- (\gamma \partial_j v + D_y(\partial_j v)[y])\mu_\epsilon' - D_y v d_\epsilon' - \mu_\epsilon'(\gamma v + D_y v[y]) + \mu_\epsilon \partial_j v\Big] \\
&+ O\left(|\mu_\epsilon \bar{y} + \bar{d}_\epsilon|^2 + (\mu_\epsilon y_N + d_{\epsilon,N})(\mu_\epsilon \bar{y} + \bar{d}_\epsilon) + (\mu_\epsilon y_N + d_{\epsilon,N})^2\right) \mu_\epsilon \partial_N v.
\end{aligned} \tag{3.16}
$$

In the above expressions (3.10)-(3.16), R_{ikjl} is defined in (2.1), $\bar{h}_{ij}$ is defined in (2.3), $\bar{k}$ is defined in (2.4), and a^{ij}_{Nk} is a smooth function of ρy_0 with

$$A^{ij}_{Nk} = a^{ij}_{Nk} x_N + O(x_N^2),$$

where A^{ij}_{Nk} is defined in (2.5). b^0_k is a smooth function of ρy_0 with

$$B^0_k = b^0_k x_N + O(x_N^2),$$

where B^0_k is defined in (2.5). b^j_N is a smooth function of ρy_0 with

$$B^j_N = b^j_N x_N + O(x_N^2),$$

where B^0_k is defined in (2.5).

After performing the change of variable (3.1) under the local coordinates along Γ, the original equation becomes

$$\mathcal{A} v + \mu_\epsilon^{\frac{N-2}{2}\epsilon} v^{\frac{N+2}{N-2}-\epsilon} = 0.$$

Define the error of w by

$$S_\epsilon(w) = \mathcal{A} w + \mu_\epsilon^{\frac{N-2}{2}\epsilon} w^{\frac{N+2}{N-2}-\epsilon}.$$

3.3. **The first approximate solution.** We first define a smooth cut-off function

$$\chi(s) = \begin{cases} 1, & \text{if } s < \hat{\delta} \\ 0, & \text{if } s > 2\hat{\delta} \end{cases}$$

and

$$\chi_\epsilon(y) = \chi(\epsilon^{\frac{1}{N-2}}|y|),$$

where $\hat{\delta}$ in (3.9) is chosen such that

$$\chi_\epsilon(\bar{y}, -\frac{d_{\epsilon,N}}{\mu_\epsilon}) = 0. \tag{3.17}$$

Denote Z_0 by the eigenfunction corresponding to the only negative eigenvalue λ_0 of the following eigenvalue problem

$$\Delta_y \phi + p w(y)^{p-1}\phi + \lambda\phi = 0,\ \phi \in L^\infty(\mathbb{R}^N),$$

namely,

$$\Delta_y Z_0 + p w(y)^{p-1} Z_0 + \lambda_0 Z_0 = 0 \tag{3.18}$$

with $\lambda_0 < 0$.

Our first approximate solution close to the geodesic Γ is

$$\mathtt{w} = \tilde{w} + e_\epsilon(\rho y_0)\chi_\epsilon(y) Z_0, \tag{3.19}$$

where $\tilde{w}$ is defined by

$$\tilde{w}(y) = (1+\alpha_\epsilon)[w(y) - \bar{w}(y)],$$

with the Aubin-Talenti bubble w, $\alpha_\epsilon := \mu_\epsilon^{-\frac{(N-2)^2\epsilon}{8}} - 1$ and

$$\bar{w}(y) = w(\bar{y}, y_N + \frac{2d_{\epsilon,N}}{\mu_\epsilon}).$$

Observe that $(1+\alpha_\epsilon)w$ satisfies

$$\Delta[(1+\alpha_\epsilon)w] + \mu_\epsilon^{\frac{N-2}{2}\epsilon}[(1+\alpha_\epsilon)w]^{\frac{N+2}{N-2}} = 0 \quad \text{in } \mathbb{R}^N,$$

and $\tilde{w} = 0$ on $y_N = -\frac{d_{\epsilon,N}}{\mu_\epsilon}$. In (3.19), $e_\epsilon(\rho y_0)$ is defined as

$$e_\epsilon = \epsilon \tilde{e}_\epsilon \tag{3.20}$$

with

$$\tilde{e}_\epsilon = e_\epsilon^0 + \epsilon e \;\text{ and }\; e_\epsilon^0 = e_0 + \epsilon^{\frac{1}{N-2}} e_1, \tag{3.21}$$

where e_1 is a smooth function uniformly bounded in ϵ, and

$$e_0 = \frac{2\int_{\mathbb{R}^N} \partial_{ii} w Z_0}{|\lambda_0|}(\text{Tr}_{\bar{g}}\bar{h} - \bar{h}_{00}) d_{0,N}. \tag{3.22}$$

The purpose of the parameter function e is to eliminate the resonance which can produce large noise in the tangential direction. We shall choose e in the final section and the norm of e is defined as

$$\|e\|_e = \|\epsilon^{2+\frac{2}{N-2}} e''\|_\infty + \|\epsilon^{1+\frac{1}{N-2}} e'\|_\infty + \|e\|_\infty. \tag{3.23}$$

3.4. **Error of the first approximate solution** $\mathtt{w}$**.** Suppose that our parameter functions μ, d and e satisfy

$$\|(\mu, d, e)\| := \|\mu\|_a + \|d\|_d + \|e\|_e \le c, \tag{3.24}$$

where the definitions of the above norms are given in (3.5), (3.6) and (3.23), $c > 0$ is a constant independent of ϵ.

By a similar computation as in [18, (5.33)], the expansion of the error $S_\epsilon(\mathtt{w})$ for small ϵ is given by

$$\begin{aligned}
S_\epsilon(\mathtt{w}) =& - pw^{p-1}\bar{w} - \epsilon w^p \log w + \epsilon(-2\bar{h}_{ij} d^0_{\epsilon,N}\partial_{ij} w + |\lambda_0| e^0_\epsilon Z_0)\\
&+ \epsilon^{\frac{N-1}{N-2}}\mu^0_\epsilon(-2\bar{h}_{ij} y_N \partial_{ij} w + \mathrm{Tr}_{\bar{g}}\bar{h}\partial_N w)\\
&+ \epsilon^2\Big[(\rho^2 a_0 e'' + |\lambda_0| e) Z_0 - 2\bar{h}_{ij} d_N \partial_{ij} w\\
&+ \sum_{i,j}(d'_i d'_j - \frac{1}{3}R_{ijkl} d_k d_l + a^{ij}_{Nk} d_k d^0_{\epsilon,N} + 4\bar{h}_{0j} d_i d^0_{\epsilon,N})\partial_{ij} w + \Upsilon_\epsilon\Big]\\
&+ \epsilon^{\frac{2N-3}{N-2}}\mu^0_\epsilon\Big[- \sum_j \partial_j w\cdot d''_j + (-\sum_{i,j}\frac{1}{3}R_{ijkl} y_k d_l \partial_{ij} w + 2a^{ij}_{Nk} y_k d^0_{\epsilon,N}\partial_{ij} w)\\
&+ (\frac{2}{3}R_{ijik} + R_{0j0k}) d_k \partial_j w + 4\bar{h}_{0j} d'_i y_N \partial_{ij} w\Big]\\
&+ \epsilon^{\frac{3N-5}{N-2}}\Big[- \mu^0_\epsilon \partial_N w\cdot d''_N - \frac{1}{3}\mu^0_\epsilon R_{ijkl} y_k d_l \partial_{ij} w + \mu\left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) d_k \partial_j w\\
&+ (\mu^0_\epsilon d_N + \mu d^0_{\epsilon,N})(2a^{ij}_{Nk} y_k \partial_{ij} w + b^j_N \partial_j w - \mathrm{Tr}_{\bar{g}}\bar{h}\partial_N w)\\
&+ (\mu^0_\epsilon e + \mu e^0_\epsilon)(-2\bar{h}_{ij} y_N \partial_{ij} Z_0 + \mathrm{Tr}_{\bar{g}}\bar{h}\partial_N Z_0)\Big]\\
&+ \epsilon^{\frac{3N-4}{N-2}}\Bigg[- \mu''\mu Z_{N+1} + 2\mu\mu^0_\epsilon\Big(- \frac{1}{3}R_{ikjl} y_k y_l \partial_{ij} w\\
&+ \left(\frac{2}{3}R_{ijik} + R_{0j0k}\right) y_k \partial_j w + b^j_N y_N \partial_j w - \mathrm{Tr}_{\bar{g}}\bar{k} y_N \partial_N w\Big)\Bigg]\\
&+ \epsilon^4(\log\epsilon) r,
\end{aligned} \tag{3.25}$$

where

$$\Upsilon_\epsilon = \Upsilon_0 + \epsilon^{\frac{1}{N-2}}\Upsilon^1_\epsilon$$

with

$$\Upsilon_0 = -2\bar{h}_{ij} d_{0,N} e_0 \partial_{ij} Z_0 + p(p-1) e_0^2 w^{p-2} Z_0^2 + p e_0 w^{p-1}\log w Z_0,$$

and Υ^1_ϵ is a smooth function of the form

$$f_1(\rho y_0) f_2(\mu, d, e) f_3(y).$$

In the above expression, f_1 is smooth and uniformly bounded in ϵ, f_2 is smooth and uniformly bounded in ϵ and f_3 is smooth with

$$\sup_{\mathcal{D}}\langle y\rangle^{N-2}|f_3(y)| < +\infty.$$

Note that f_2 depends linearly on μ'', d'' and e''. We refer the reader to [18, Appendix] for the detailed computations of (3.25).

From (3.25), we can write

$$S_\epsilon(\mathtt{w}) = \epsilon S_0 + \epsilon^2(\rho^2 a_0 e'' + |\lambda_0| e)\chi_\epsilon Z_0 + \epsilon^2 S_1, \tag{3.26}$$

where

$$\begin{aligned}
\epsilon S_0(\rho y_0) :=& - pw^{p-1}\bar{w} - \epsilon w^p \log w + \epsilon(-2\bar{h}_{ij} d^0_{\epsilon,N}\partial_{ij} w + |\lambda_0| e^0_\epsilon Z_0)\\
&+ \epsilon^{\frac{N-1}{N-2}}\mu^0_\epsilon(-2\bar{h}_{ij} y_N \partial_{ij} w + \mathrm{Tr}_{\bar{g}}\bar{h}\partial_N w),
\end{aligned}$$

S_0 is smooth and uniformly bounded in ϵ, and S_1 depends on μ, d and e.

3.5. **Correction of the approximate solution.** Now we add an extra correction Π to get rid of ϵS_0 in (3.26), namely, Π solves the following linear problem

$$\begin{cases} a_0\partial_0^2\Pi + \Delta_y\Pi + \tilde{\mathcal{A}}\Pi + pw^{p-1}\Pi = -\epsilon S_0 + \sum_{i=0}^{N+1}\alpha_i Z_i \text{ in } \mathcal{D} \\ \Pi(y_0,y) = 0 \text{ on } \partial\mathcal{D}_{y_0} \text{ for all } y_0, \end{cases}$$

where

$$\mathcal{D}_{y_0} := \{y \in \mathbb{R}^N : (y_0,y) \in \mathcal{D}\}$$

with $\mathcal{D}$ defined in (3.9). By choosing suitable parameters μ, d and e at main order, namely μ_ϵ^0, $d_{\epsilon,N}^0$ and e_ϵ^0 (see (3.3), (3.21) and (3.22) for their definitions), the orthogonality conditions

$$\int_{\mathcal{D}_{y_0}} S_0 Z_i dy = 0, \text{ for all } y_0 \text{ and } i = 0,1,\cdots,N+1,$$

are achieved. The argument is the same as that of [18, Appendix]. We omit the details.

We only need to consider the leading term in ϵS_0

$$h_0 = -pw^{p-1}\bar{w} - \epsilon w^p \log w + \epsilon(-2\bar{h}_{ij}d_{\epsilon,N}^0\partial_{ij}w + |\lambda_0|e_\epsilon^0 Z_0),$$

since other terms are of sufficiently fast decay. Note that

$$|w^p(y)| \lesssim \frac{1}{1+|y|^{N+2}},$$

$$\left|-2\bar{h}_{ij}d_{\epsilon,N}^0\partial_{ij}w + |\lambda_0|e_\epsilon^0 Z_0\right| \lesssim \frac{1}{1+|y|^N},$$

and

$$|w^{p-1}\bar{w}|(y_0,y) \lesssim \frac{1}{1+|y|^4}\frac{1}{1+|\bar{y}|^{N-2}+(y_N+\epsilon^{-\frac{1}{N-2}})^{N-2}}.$$

In order to apply the linear theory as in [18, Section 3], we can gain enough dacay in y by losing a little bit ϵ. To be more precise, for fixed $\vartheta > 0$ close to 0, it holds that

$$|w^{p-1}\bar{w}| = |(w^{p-1}\bar{w}^{\vartheta})\bar{w}^{1-\vartheta}| \lesssim \frac{\epsilon^{1-\vartheta}}{1+|y|^{4+(N-2)\vartheta}},$$

in $\mathcal{D}$. We consider the problem

$$\begin{cases} a_0\partial_0^2\Pi + \Delta_y\Pi + \tilde{\mathcal{A}}\Pi + pw^{p-1}\Pi = -\epsilon S_0 + \sum_{i=0}^{N+1}\alpha_i Z_i \text{ in } \mathcal{D} \\ \Pi(y_0,y) = 0 \text{ on } \partial\mathcal{D}_{y_0} \text{ for all } y_0 \\ \int_{\mathcal{D}_{y_0}}\Pi(y_0,y)Z_i dy = 0, \text{ for all } y_0 \text{ and } i = 0,1,\cdots,N+1. \end{cases} \tag{3.27}$$

Since

$$\|\partial_0(\frac{d_{\epsilon,N}}{\mu_\epsilon})\|_\infty \lesssim \rho\epsilon^{-\frac{1}{N-2}}(\epsilon\|\mu'\|_\infty + \epsilon\|d_N'\|_\infty) = o(1)$$

and

$$\rho^{-1}\|\partial_{00}(\frac{d_{\epsilon,N}}{\mu_\epsilon})\|_\infty \lesssim \rho\epsilon^{-\frac{1}{N-2}}(\epsilon\|\mu''\|_\infty + \epsilon\|d_N''\|_\infty) = o(1)$$

as $\epsilon \to 0^+$, problem (3.27) satisfies assumptions (5.3) and (5.4) in Proposition 5.1. Therefore, from Proposition 5.1, there exists a unique tuple (α_i, Π) solving the elliptic problem (3.27). Moreover, the solution Π satisfies

$$|\Pi(y_0, y)| \lesssim \frac{\epsilon^{1-\vartheta}}{1+|y|^{2+(N-2)\vartheta}}, \quad \forall (y_0, y) \in \mathcal{D}, \tag{3.28}$$

namely

$$\|\Pi\|_\sigma \lesssim \epsilon^{1-\vartheta},$$

where we have used the facts that $0 < \sigma \le N-4$ and $N = 5, 6$. Since Π only depends on μ and d, by Lemma 3.1, we can estimate

$$\|\Pi_{\mu_1,d_1} - \Pi_{\mu_2,d_2}\|_\sigma \le c\epsilon^{2-\vartheta}\|(\mu_1-\mu_2, d_1-d_2)\|.$$

Moreover, using similar computations as in [18, (5.47)], we have

$$\sup |\alpha_i| \le o(1)\epsilon^3. \tag{3.29}$$

Let $\psi = \partial_0 \Pi$. Then by differentiating the equation (3.27) with respect to y_0, we have

$$\begin{cases} a_0\partial_0^2\psi + \Delta_y\psi + \tilde{\mathcal{A}}\psi + pw^{p-1}\psi + \rho a_0'\partial_0\psi = h \text{ in } \mathcal{D} \\ \int_{\mathcal{D}_{y_0}} \psi(y_0, y)Z_i dy = 0 \text{ for all } y_0 \text{ and } i = 0, 1, \cdots, N+1 \\ \psi(y_0, \bar{y}, y_N) - \partial_0\left(\frac{d_{\epsilon N}}{\mu_\epsilon}\right)\partial_N\Pi(y_0, \bar{y}, y_N) = 0 \text{ on } \partial\mathcal{D}_{y_0} \text{ for all } y_0, \end{cases}$$

where $h = -\epsilon\rho\partial_0 S_0 - (\partial_0\tilde{\mathcal{A}})\Pi + \sum_{i=0}^{N+1}\partial_0\alpha_i Z_i$. Then by a similar argument as above, we find that

$$\|\psi\|_\sigma \lesssim \rho\epsilon^{1-\vartheta}.$$

Therefore, with the correction Π which eliminates the term ϵS_0 in (3.26), the new error for our new approximate solution

$$\mathtt{W} = \mathtt{w} + \Pi$$

is the following

$$S_\epsilon(\mathtt{W}) = \epsilon^2 S_1 + \epsilon^2(\rho^2 a_0 e'' + |\lambda_0|e)\chi_\epsilon Z_0 + N_1(\Pi) + \sum_{i=0}^{N+1}\alpha_i Z_i, \tag{3.30}$$

where

$$N_1(\Pi) = \mu_\epsilon^{-\frac{N-2}{2}\epsilon}[(\mathtt{w}+\Pi)^{p-\epsilon} - \mathtt{w}^{p-\epsilon}] - pw^{p-1}\Pi. \tag{3.31}$$

The dependence of S_1 on the parameter function μ, d and e is given by

$$\|S_1(\mu_1, d_1, e_1) - S_1(\mu_2, d_2, e_2)\|_{2+\sigma} \le c\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

Further, by (3.28), we know that $w \gtrsim |\Pi|$ gives

$$|y| \lesssim \epsilon^{\frac{\vartheta-1}{N-4-(N-2)\vartheta}}$$

and vice versa. By the definition (1.8) and $R = R(\epsilon) = \epsilon^{-\theta_*}$ with θ_* given in (7.13), we see that

$$R \ll \epsilon^{\frac{\vartheta-1}{N-4-(N-2)\vartheta}},$$

where we have used $\vartheta \approx 0$. Thus by a direct Taylor expansion, $\|N_1(\Pi)\|_{2+\sigma}$ can be estimated as follows

$$\begin{aligned}\|N_1(\Pi)\|_{2+\sigma} &= \sup_{\mathbb{S}_\rho\times D_R} \langle y\rangle^{2+\sigma}|N_1(\Pi)| \lesssim \sup_{|y|\lesssim \epsilon^{\frac{\vartheta-1}{N-4-(N-2)\vartheta}}} \langle y\rangle^{2+\sigma} w^{p-2}\Pi^2 \\ &= \sup_{|y|\lesssim \epsilon^{\frac{\vartheta-1}{N-4-(N-2)\vartheta}}} \langle y\rangle^{2+\sigma}\langle y\rangle^{N-6}\langle y\rangle^{-4-2(N-2)\vartheta}\epsilon^{2(1-\vartheta)} \\ &\lesssim \epsilon^{2(1-\vartheta)}.\end{aligned} \tag{3.32}$$

This finishes the construction of the approximate solution W and the estimate of the new error $S_\epsilon(\mathtt{W})$.

4. The inner-outer gluing procedure

In this section, we shall apply the inner-outer gluing scheme to find a true solution based on the approximate solution W we built in Section 3.

Letting $u(z) = \rho^{-\frac{N-2}{2}} v(\frac{z}{\rho})$, equation (1.4) becomes

$$\begin{cases}\Delta v + \rho^{\frac{(N-2)\epsilon}{2}} v^{p-\epsilon} = 0, & \text{in } \Omega_\rho\\ v>0, & \text{in } \Omega_\rho\\ v=0, & \text{on } \partial\Omega_\rho\end{cases}$$

where $p = \frac{N+2}{N-2}$ and $\Omega_\rho = \frac{1}{\rho}\Omega$. With a slight abuse of notation, we replace v by u, namely,

$$\begin{cases}\Delta u + \rho^{\frac{(N-2)\epsilon}{2}} u^{p-\epsilon} = 0, & \text{in } \Omega_\rho\\ u>0, & \text{in } \Omega_\rho\\ u=0, & \text{on } \partial\Omega_\rho\end{cases}$$

In this problem, we have local coordinates near the geodesic Γ and the corresponding Laplace-Beltrami operator encodes the geometric information of the geodesic Γ, while in the region far away from the geodesic Γ, we use the usual Euclidean coordinates. Therefore, after introducing some suitable cut-off functions, it is natural to decompose the full problem into the inner and outer parts in which the inner-outer gluing procedure can be carried out.

4.1. The global approximate solution.

In a small neighborhood of the geodesic Γ, we denote

$$f(z) = \tilde{\mu}_\epsilon^{-\frac{N-2}{2}}(\rho y_0)\tilde{f}(y_0,y), \text{ where } z = \frac{1}{\rho}G(\rho y_0, \rho\tilde{\mu}_\epsilon(\rho y_0)y + \epsilon\tilde{d}_\epsilon(\rho y_0)),$$

or equivalently,

$$\tilde{f}(y_0,y) = \tilde{\mu}_\epsilon^{\frac{N-2}{2}}(\rho y_0) f(z),$$

where $z\in\mathbb{R}^{N+1}$ is the original variable in Ω_ρ. Near the geodesic, the approximate solution we construct in previous section W now becomes $\tilde{\mathbf{w}}$ in this setting. Indeed, recall that after the change of variables (3.1), we have

$$\mathcal{A}v + \mu_\epsilon^{\frac{(N-2)\epsilon}{2}} v^{p-\epsilon} = 0.$$

Observe that $\tilde{\mathbf{w}}$ is only defined locally on a small neighborhood near the geodesic. In order to get a global approximate solution, we first introduce some cut-off functions. Let $\delta>0$ be a fixed number with $4\delta < \hat{\delta}$, where $\hat{\delta}$ is chosen in

(3.17). Consider a standard cut-off function $\zeta_\delta(s)$ with $\zeta_\delta(s) = 1$ for $0 < s < \delta$ and $\zeta_\delta(s) = 0$ for $s > 2\delta$. We define

$$\zeta_\delta^\epsilon(y_0, y) = \zeta_\delta(|G(\rho y_0, \tilde{\mu}_\epsilon(\rho y_0)\rho y + \epsilon \tilde{d}_\epsilon(\rho y_0))|),$$

and

$$\eta_{\delta,2R}^\epsilon(z) = \zeta_\delta^\epsilon(y_0, y/R(\epsilon)),$$

where

$$R(\epsilon) = \epsilon^{-\theta_*} \text{ with } 0 < \theta_* < 1 \tag{4.1}$$

which we will specify later. For simplicity, we write $R(\epsilon)$ as R in the sequel. By a zero extension of $\mathtt{W}$ far away from $\frac{1}{\rho}\Gamma$, it is natural to choose our first global approximate solution as

$$\mathbf{w}(z) = \eta_{\delta,2R}^\epsilon(z)\tilde{\mathbf{w}}(z).$$

We look for a solution $u = \mathbf{w} + \Phi$ where $\Phi = \eta_{2\delta,2R}\phi + \psi$ and ϕ is such that $\tilde{\phi}$ is in principle defined only in $\mathcal{D}$. Here $\mathcal{D}$ is defined in (3.9). Then Φ satisfies

$$\begin{cases} \Delta\Phi + p\mathbf{w}^{p-1}\Phi + N(\Phi) + E = 0, & \text{in } \Omega_\rho \\ \Phi = 0, & \text{on } \partial\Omega_\rho \end{cases} \tag{4.2}$$

where

$$N(\Phi) = \rho^{\frac{(N-2)\epsilon}{2}}(\mathbf{w} + \Phi)^{p-\epsilon} - \mathbf{w}^{p-\epsilon} - p\mathbf{w}^{p-1}\Phi \quad \text{and} \quad E = \Delta\mathbf{w} + \mathbf{w}^{p-\epsilon}. \tag{4.3}$$

Recall that near the stretched geodesic $\Gamma_\rho = \frac{1}{\rho}\Gamma$, the error is

$$S_\epsilon(\tilde{\mathbf{w}}) = \mathcal{A}\tilde{\mathbf{w}} + \mu_\epsilon^{\frac{(N-2)\epsilon}{2}}\tilde{\mathbf{w}}^{p-\epsilon}.$$

Then near Γ_ρ, $S_\epsilon(\tilde{u}) = 0$ implies that the equation for $\tilde{\Phi}$ is

$$\mathcal{A}\tilde{\Phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\Phi} + \mathtt{N}(\tilde{\Phi}) + S_\epsilon(\tilde{\mathbf{w}}) = 0,$$

where

$$\mathtt{N}(\tilde{\Phi}) = \mu_\epsilon^{\frac{(N-2)\epsilon}{2}}(\tilde{\mathbf{w}} + \tilde{\Phi})^{p-\epsilon} - \mu_\epsilon^{\frac{(N-2)\epsilon}{2}}\tilde{\mathbf{w}}^{p-\epsilon} - p\tilde{\mathbf{w}}^{p-1}\tilde{\Phi}.$$

4.2. **Inner and outer problems.** It is direct to see that $\Phi = \eta_{2\delta,2R}\phi + \psi$ solves (4.2) if the tuple $(\tilde{\phi}, \psi)$ solves the following two coupled equations

- $\tilde{\phi}$ solves the so-called **inner problem**

$$\begin{cases} \mathcal{A}\tilde{\phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\phi} = -\mathtt{N}(\zeta_{2\delta}^\epsilon\tilde{\phi} + \tilde{\psi}) - S_\epsilon(\tilde{\mathbf{w}}) - p\tilde{\mathbf{w}}^{p-1}\tilde{\psi}, & \text{in } \mathbb{S}_\rho \times D_R \\ \tilde{\phi} = 0, & \text{on } \partial(\mathbb{S}_\rho \times D_R) \end{cases} \tag{4.4}$$

- ψ solves the so-called **outer problem**

$$\begin{cases} \Delta\psi = -(1 - \eta_{\delta,2R}^\epsilon)p\mathbf{w}^{p-1}\psi & -2\nabla\phi\cdot\nabla\eta_{\delta,2R}^\epsilon - \phi\Delta\eta_{\delta,2R}^\epsilon \\ & -(1 - \eta_{\delta,2R}^\epsilon)N(\eta_{\delta,2R}^\epsilon\phi + \psi), \quad \text{in } \Omega_\rho \\ \psi = 0, \quad \text{on } \partial\Omega_\rho \end{cases} \tag{4.5}$$

Define the following norm

$$\|\phi\|_* := \sup_{\mathbb{S}_\rho \times D_R} R^{\sigma-\tau}\langle y\rangle^\tau |\phi(y_0, y)| \tag{4.6}$$

with $0 < \sigma \le N - 4$ and $2 < \tau < N - 2$.

We will first solve the outer problem (4.5) for given $\tilde{\phi}$ with sufficiently small $\|\cdot\|_*$-norm. After we get the outer solution $\psi(\phi)$, the inner problem (4.4) is reduced to a nonlinear nonlocal problem. In order to solve the reduced inner problem, we shall develop a linear theory for the solvability of the associated model problem of the projected inner problem. Then by applying the linear theory and the Contraction Mapping Theorem, we solve the projected inner problem. Finally, we shall adjust our parameter functions μ, d and e such that the reduced system $c_j(\rho y_0) = 0$ is satisfied for all y_0 and $j = 0, 1, \cdots, N+1$.

4.3. **Solving the outer problem.** Recall the definition of $\|\cdot\|_*$ in (4.6). Given ϕ such that $\|\tilde{\phi}\|_* \lesssim \epsilon^{2(1-\vartheta)}$ in $\mathcal{D}$, we solve the outer problem (4.5) first.

Case 1. Assume that Ω is bounded. Then the following simple problem

$$\begin{cases} -\Delta\psi = h_1, & \text{in } \Omega_\rho \\ \psi = 0, & \text{on } \partial\Omega_\rho \end{cases}$$

has a unique solution $\psi = (-\Delta)^{-1}h_1$ for $h_1 \in L^\infty(\Omega_\rho)$. Further, we have

$$\|\psi\|_\infty \lesssim \|h_1\|_\infty.$$

Now we consider each term on the right hand side of the first equation of (4.5). Due to the effect of cut-off functions, we obtain from (4.6) that

$$\|\tilde{\phi}\|_{L^\infty(R<|y|<2R)} \sim R^{-\sigma}\|\tilde{\phi}\|_*$$

and

$$\|\phi\Delta\eta^\epsilon_{\delta,2R}\|_\infty \lesssim \frac{1}{R^2}\|\tilde{\phi}\|_{L^\infty(R<|y|<2R)} \lesssim \frac{1}{R^{2+\sigma}}\|\tilde{\phi}\|_*. \tag{4.7}$$

Similarly, we have

$$\|\nabla\phi\cdot\nabla\eta^\epsilon_{\delta,2R}\|_\infty \lesssim \frac{1}{R^{2+\sigma}}\|\tilde{\phi}\|_*. \tag{4.8}$$

According to decay of ϕ, we assume $\|\psi\|_\infty \le \Lambda R^{-\sigma}\|\tilde{\phi}\|_*$ with Λ fixed sufficiently large and set

$$M(\psi) := (1-\eta^\epsilon_{\delta,2R})N(\eta^\epsilon_{\delta,2R}\phi + \psi),$$

where $N(\eta^\epsilon_{\delta,2R}\phi+\psi)$ is defined in (4.3). Then for ϵ small, we have

$$\|M(\psi)\|_\infty \lesssim (1+\Lambda^p)\frac{1}{R^{\frac{(N+2)\sigma}{N-2}}}\|\phi\|_*^p. \tag{4.9}$$

Moreover,

$$\|(1-\eta^\epsilon_{\delta,2R})p\mathbf{w}^{p-1}\psi\|_\infty \lesssim \frac{1}{R^4}\|\psi\|_\infty. \tag{4.10}$$

By the Contraction Mapping Theorem, for R sufficiently large (by (4.1) this is possible as ϵ is small enough), the fixed point problem

$$\begin{aligned} &\psi = \mathcal{T}(\psi) \\ &:= (-\Delta)^{-1}\left(M(\psi) + (1-\eta^\epsilon_{\delta,2R})p\mathbf{w}^{p-1}\psi + \phi\Delta\eta^\epsilon_{\delta,2R} + 2\nabla\phi\cdot\nabla\eta^\epsilon_{\delta,2R}\right) \end{aligned} \tag{4.11}$$

has a unique solution $\psi = \psi(\phi)$ in the function space $\mathcal{N} = \{\psi : \|\psi\|_\infty \le \Lambda R^{-\sigma}\|\tilde{\phi}\|_*\}$ provided $\|\tilde{\phi}\|_* \lesssim \epsilon^{2(1-\vartheta)}$. Indeed, from (4.7)-(4.11) we have that

$$\|\mathcal{T}(\psi)\|_\infty \lesssim R^{-\sigma}\|\phi\|_* \quad \text{for } N = 5, 6, \tag{4.12}$$

where we have used the assumptions $\|\phi\|_* \lesssim \epsilon^{2(1-\vartheta)}$, R is sufficiently large and ϵ is small. Therefore, the mapping $\mathcal{T}(\psi)$ maps $\mathcal{N}$ to itself. On the other hand, from (4.9), we see that for $\psi^{(1)}, \psi^{(2)} \in \mathcal{N}$

$$\begin{aligned}\|M(\psi^{(1)}) - M(\psi^{(2)})\|_\infty &\lesssim \left(\|\tilde{\phi}\|_{L^\infty(R<|y|<2R)} + \Lambda \frac{1}{R^\sigma}\|\tilde{\phi}\|_*\right)^{p-1} \|\psi^{(1)} - \psi^{(2)}\|_\infty \\ &\lesssim (1+\Lambda)^{\frac{4}{N-2}} \frac{1}{R^{\frac{4\sigma}{N-2}}} \|\tilde{\phi}\|_*^{\frac{4}{N-2}} \|\psi^{(1)} - \psi^{(2)}\|_\infty.\end{aligned} \tag{4.13}$$

Therefore, from (4.10) and (4.13) we have

$$\|\mathcal{T}(\psi^{(1)}) - \mathcal{T}(\psi^{(2)})\|_\infty \lesssim \left[(1+\Lambda)^{\frac{4}{N-2}} \frac{1}{R^{\frac{4\sigma}{N-2}}} \|\tilde{\phi}\|_*^{\frac{4}{N-2}} + \frac{1}{R^4}\right] \|\psi^{(1)} - \psi^{(2)}\|_\infty,$$

which implies $\mathcal{T}$ is a contraction mapping in the space $\mathcal{N}$ for ϵ sufficiently small and R sufficiently large. Thus, the Contraction Mapping Theorem implies the existence of such $\psi \in \mathcal{N}$.

Moreover, from (4.11) and (4.12), the Lipschitz dependence of ψ on ϕ is given by

$$\|\psi(\phi_1) - \psi(\phi_2)\|_\infty \lesssim R^{-\sigma}\|\phi_1 - \phi_2\|_* \ \text{ for } N = 5, 6.$$

Case 2. Next we consider the unbounded case and let $\Omega = \mathbb{R}^{N+1} \backslash \Upsilon$ with Υ bounded.

Observe that the coupling term $-2\nabla\phi \cdot \nabla\eta^\epsilon_{\delta,2R} - \phi\Delta\eta^\epsilon_{\delta,2R}$ in the outer problem (4.5) is supported in $\mathbb{S}_\rho \times (D_R \backslash D_{R/2})$, where

$$D_R \backslash D_{R/2} = \{y \in \mathbb{R}^N : R < |y| < 2R\}.$$

So we decompose the outer problem into the following two equations

$$\begin{cases} \Delta\psi_1 = -2\nabla\phi \cdot \nabla\eta^\epsilon_{\delta,2R} - \phi\Delta\eta^\epsilon_{\delta,2R}, & \text{in } \mathbb{S}_\rho \times (D_R \backslash D_{R/2}) \\ \psi_1 = 0, \quad \text{on } \partial\mathbb{S}_\rho \times (D_R \backslash D_{R/2}) \end{cases} \tag{4.14}$$

and

$$\begin{cases} \Delta\psi_2 = -(1-\eta^\epsilon_{\delta,2R}) p\mathbf{w}^{p-1}\psi_2 - (1-\eta^\epsilon_{\delta,2R}) N(\eta^\epsilon_{\delta,2R}\phi + \psi), & \text{in } \Omega_\rho \\ \psi_2 = 0, \quad \text{on } \partial\Omega_\rho. \end{cases} \tag{4.15}$$

For equation (4.14), by a same argument as in the **Case** 1, we obtain that there exists solution $\psi_1 \in \mathcal{N}$, namely,

$$\|\psi_1\|_\infty \lesssim R^{-\sigma}\|\tilde{\phi}\|_* \tag{4.16}$$

.

We pull back the equation (4.15) for ψ_2 from Ω_ρ to Ω. Define $\hat{f}(z) = f(\frac{z}{\rho})$. Then the equation (4.15) becomes

$$\begin{cases} \Delta\hat{\psi}_2 = -\rho^{-2}(1-\hat{\eta}^\epsilon_{\delta,2R}) p\hat{\mathbf{w}}^{p-1}\hat{\psi}_2 - \rho^{-2}(1-\hat{\eta}^\epsilon_{\delta,2R})(\hat{\eta}^\epsilon_{\delta,2R}\hat{\phi} + \hat{\psi})^p & \text{in } \Omega \\ \hat{\psi}_2 = 0 & \text{on } \partial\Omega \end{cases}$$

namely,

$$\begin{cases} \Delta\hat{\psi}_2 = -\frac{O(\rho^2)(1-\chi)\hat{\psi}_2}{\rho^4+|z|^4} - \rho^{-2}(1-\chi)\left(O(R^{-\sigma})\|\tilde{\phi}\|_*\chi + \hat{\psi}\right)^p & \text{in } \Omega, \\ \hat{\psi}_2 = 0 & \text{on } \partial\Omega \end{cases} \tag{4.17}$$

where χ is a smooth function with compact support. In the exterior domain, after a perform of Kelvin transform with respect to a given point q in the interior $\mathring{\Upsilon} \subset \mathbb{R}^{N+1} = \mathbb{R}^n$

$$K\hat{\psi}_2(x) = |x-q|^{2-(N+1)}\hat{\psi}_2(z) \ \text{ with } z = \frac{x-q}{|x-q|^2} \in \Omega \ \text{ and } q \in \mathring{\Upsilon},$$

we see that the equation (4.17) becomes the following equation in the bounded domain $K(\Omega)$

$$\begin{cases} \Delta(K\hat{\psi}_2(x)) = -\dfrac{O(\rho^2)(1-\chi_1)}{1+|x-q|^4}K\hat{\psi}_2(x) - \dfrac{1}{\rho^2|x-q|^{N+3}}\Big[O(R^{-\sigma})\|\tilde{\phi}\|_*\chi_3 \\ \qquad\qquad + |x-q|^{N-1}K\hat{\psi}(1-\chi_4)\Big]^p, \ \text{ in } K(\Omega) \\ K\hat{\psi}_2(x) = 0, \ \text{ on } \partial K(\Omega) \end{cases} \tag{4.18}$$

which has a solution $K\hat{\psi}_2 = (-\Delta)^{-1}h_2$ with

$$\|K\hat{\psi}_2(x)\|_\infty \lesssim \|h_2(x)\|_\infty < +\infty, \ x \in K(\Omega),$$

namely

$$\Big\||z|^{N-1}\hat{\psi}_2\,(z)\Big\|_\infty \lesssim \|h_2(x)\|_\infty < +\infty, \ x \in K(\Omega), \tag{4.19}$$

where h_2 is the nonhomogeneous term in (4.18). Here χ_1 and χ_4 are cut-off functions supported in neighborhood of the geodesic Γ, χ_2 and χ_3 are two cut-off functions supported in bounded annulus. If $\|\hat{\psi}_2\|_\infty \le \Lambda R^{-\sigma}\|\tilde{\phi}\|_*$, we see that

$$\|h_2(x)\|_\infty \sim \max\{\rho^2 R^{-\sigma}\|\tilde{\phi}\|_*, \rho^{-2}R^{-\sigma p}\|\tilde{\phi}\|_*^p\} \tag{4.20}$$

for ϵ small. Since $K(\Omega)$ is bounded and $\|\tilde{\phi}\|_* \lesssim \epsilon^{2(1-\vartheta)}$ with $\vartheta > 0$ close to 0, from (4.20) we get

$$\|h_2\|_\infty \lesssim R^{-\sigma}\|\tilde{\phi}\|_*, \quad \text{ for } N = 5, 6,$$

where θ_* cannot be chosen too small. For example, in the case $N = 6$, direct computation shows that $\theta_* > 1/4$. This is valid since we will choose θ_* in (7.13) at last. By a similar fixed point argument, we can solve the problem (4.15) in the function space $\mathcal{N} = \{\psi : \|\psi\|_\infty \le \Lambda R^{-\sigma}\|\tilde{\phi}\|_*\}$ whenever $\|\tilde{\phi}\|_* \lesssim \epsilon^{2(1-\vartheta)}$ for ϵ small, and the solution ψ_2 satisfies

$$\|\psi_2\|_\infty \lesssim R^{-\sigma}\|\tilde{\phi}\|_*, \quad \text{ for } N = 5, 6. \tag{4.21}$$

Combining (4.12) and (4.21), we get that the solution $\psi = \psi_1 + \psi_2$ of the outer problem (4.5) satisfies

$$\|\psi\|_\infty \lesssim R^{-\sigma}\|\tilde{\phi}\|_*, \quad \text{ for } N = 5, 6. \tag{4.22}$$

This finishes the argument of the outer problem.

4.4. **The reduced inner problem.** As a conclusion, substituting $\tilde{\psi} = \tilde{\psi}(\tilde{\phi})$ in the inner problem (4.4), the full problem reduces to the following nonlinear nonlocal equation

$$\begin{cases} \mathcal{A}\tilde{\phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\phi} = -\mathtt{N}(\zeta_{2\delta}^\epsilon\tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - S_\epsilon(\tilde{\mathbf{w}}) - p\tilde{\mathbf{w}}^{p-1}\tilde{\psi}(\tilde{\phi}), & \text{in } \mathbb{S}_\rho \times D_R \\ \tilde{\phi} = 0, & \text{on } \partial(\mathbb{S}_\rho \times D_R) \end{cases} \tag{4.23}$$

Instead of directly solving (4.23), we shall solve the following projected problem

$$\begin{cases} \mathcal{A}\tilde{\phi} + p\tilde{\mathbf{w}}^{p-1}\tilde{\phi} = H(\tilde{\phi}, \tilde{\psi}, \mu, d, e) + \sum\limits_{j=0}^{N+1} c_j(\rho y_0) Z_j(y), & \text{in } \mathbb{S}_\rho \times D_R \\ \tilde{\phi} = 0, & \text{on } \partial(\mathbb{S}_\rho \times D_R), \end{cases} \tag{4.24}$$

where

$$H(\tilde{\phi}, \tilde{\psi}, \mu, d, e) := -\mathtt{N}(\zeta_{2\delta}^{\epsilon}\tilde{\phi} + \tilde{\psi}(\tilde{\phi})) - S_\epsilon(\tilde{\mathbf{w}}) - p\tilde{\mathbf{w}}^{p-1}\tilde{\psi}(\tilde{\phi}).$$

We shall develop a linear theory concerning the solvability for the associated model problem of (4.24) in Section 5. In Section 6, we will solve the projected inner problem (4.24) by the linear theory and the Contraction Mapping Theorem. In Section 7, we will derive and solve the reduced system of μ, d and e such that $c_j(\rho y_0) = 0$ for $j = 0, 1, \cdots, N+1$.

5. The linear theory for $N = 5$ and 6

In this section, we shall develop the linear theory concerning the existence and a priori estimates of the following linear problem

$$\begin{cases} \mathcal{A}\phi + pw^{p-1}\phi = h + \sum\limits_{j=0}^{N+1} c_j(\rho y_0) Z_j(y), \text{ in } \mathcal{D}_1 \\ \phi = 0, \text{ on } \partial\mathcal{D}_1 \end{cases} \tag{5.1}$$

in the following domain

$$\mathcal{D}_1 = \{(y_0, \bar{y}, y_N) \in \mathbb{R}^{N+1} : -\frac{d_{\epsilon,N}}{\mu_\epsilon}(\rho y_0) < y_N < M(\epsilon), |\bar{y}| < M(\epsilon)\},$$

where $M(\epsilon) > 0$ depends on ϵ. Here h satisfies

$$\|h\|_{2+\sigma} < +\infty \text{ with } 0 < \sigma \le N-4,$$

where $\|\cdot\|_{2+\sigma}$ norm is defined in (1.8) in the domain $\mathcal{D}_1$. Recall from Section 3.2 that for $(y_0, y) \in \mathcal{D}_1$,

$$\mathcal{A}v = a_0\partial_0^2 v + \Delta_y v + \tilde{A}v,$$

where

$$a_0 = \tilde{\mu}_\epsilon^2 = (\mu_0 + \epsilon^{\frac{1}{N-2}}\mu_1 + \epsilon\mu)^2,$$

and

$$\tilde{A}v = \sum_{\alpha,\beta} a_{\alpha,\beta}\partial_{\alpha,\beta}v + \sum_\alpha b_\alpha\partial_\alpha v + cv$$

with

$$a_{\alpha,\beta} = O(\epsilon + \rho^2|y|^2) = O(\epsilon) \quad \text{ for } \alpha \ne 0, \beta \ne 0,$$

$$a_{0,\beta} = O(\epsilon) \quad \text{and} \quad a_{0,0} = 0,$$

$$b_\alpha = \rho O(\epsilon + \rho|y|) = \rho O(\epsilon) \quad \text{and} \quad c = \rho^2 O(1).$$

The dimension restriction in the linear theory of [18, Section 3] is made such that the orthogonality conditions

$$\int_{\mathbb{R}^N} \phi(y_0, y) Z_j(y) dy = 0, \; j = 0, 1, \cdots, N+1$$

are well-defined. We have the following (see [18, Proposition 3.2])

Proposition 5.1. *Assume that $N = 5, 6$ and $\|\bar{h}\|_{2+\tau} < +\infty$ with $2 < \tau < N - 2$. For the linear projected problem*

$$\begin{cases} \mathcal{A}\bar{\phi} + pw^{p-1}\bar{\phi} = \bar{h} + \sum_{j=0}^{N+1} \bar{c}_j(\rho y_0) Z_j(y), \ \text{in } \mathcal{D}_1 \\ \bar{\phi} = 0, \ \text{on } \partial\mathcal{D}_1 \\ \int_{\mathcal{D}_1} \bar{\phi}(y_0, y) Z_j(y) dy = 0, \quad \text{for all } y_0 \in \mathbb{R}, j = 0, 1, \cdots, N+1, \end{cases} \tag{5.2}$$

if for all indices $\alpha, \beta = 0, 1, \cdots, N+1$,

$$\begin{aligned} \|\partial_0(\frac{d_{\epsilon,N}}{\mu_\epsilon})\|_\infty + M(\epsilon)\|\partial_{00}(\frac{d_{\epsilon,N}}{\mu_\epsilon})\|_\infty + M(\epsilon)\|\partial_0 a_0\|_\infty + \|a_{\alpha,\beta}\|_\infty + \|Da_{\alpha,\beta}\|_\infty \\ + \|\langle y\rangle b_\alpha\|_\infty + \|\langle y\rangle^2 c\|_\infty < \delta, \end{aligned} \tag{5.3}$$

and

$$\delta^{-1} < \frac{d_{\epsilon,N}}{\mu_\epsilon}(\rho y_0) < M(\epsilon)\delta \ \ \text{for all } y_0 \in \mathbb{R} \tag{5.4}$$

for some positive constant δ, then for any $\|\bar{h}\|_{2+\tau} < +\infty$ there exists a unique solution $\bar{\phi} = T(\bar{h})$ which defines a linear operator of $\bar{h}$ with $\|\bar{\phi}\|_\tau < +\infty$. Moreover, it holds that

$$\|\bar{\phi}\|_\tau \lesssim \|\bar{h}\|_{2+\tau}.$$

Denote

$$h = R^{\tau-\sigma}\bar{h} \text{ and } \phi = R^{\tau-\sigma}\bar{\phi},$$

where $2 < \tau < N - 2$. Since $\tau > \sigma$ for $N = 5, 6$, we see that

$$\|\bar{h}\|_{2+\tau} \leq \langle y\rangle^{2+\tau} R^{\sigma-\tau} \langle y\rangle^{-2-\sigma} \|h\|_{2+\sigma} \leq \|h\|_{2+\sigma}. \tag{5.5}$$

Since problem (5.2) is linear, multiplying equation (5.2) with $R^{\tau-\sigma}$ yields that ϕ solves

$$\begin{cases} \mathcal{A}\phi + pw^{p-1}\phi = h + \sum_{j=0}^{N+1} c_j(\rho y_0) Z_j(y), \ \text{in } \mathcal{D}_1 \\ \phi = 0, \ \text{on } \partial\mathcal{D}_1 \\ \int_{\mathcal{D}_1} \phi(y_0, y) Z_j(y) dy = 0, \quad \text{for all } y_0 \in \mathbb{R}, j = 0, 1, \cdots, N+1, \end{cases}$$

with $c_j(\rho y_0) = R^{\tau-\sigma}\bar{c}_j(\rho y_0)$. From Proposition 5.1 and (5.5), we obtain

$$|\phi(y_0, y)| \lesssim R^{\tau-\sigma}\langle y\rangle^{-\tau}\|h\|_{2+\sigma}.$$

The above argument concludes the following proposition.

Proposition 5.2. *Assume that $0 < \sigma \leq N - 4$, $2 < \tau < N - 2$, $N = 5, 6$ and $\|h\|_{2+\sigma} < +\infty$. If for all indices $\alpha, \beta = 0, 1, \cdots, N+1$,*

$$\begin{aligned} \|\partial_0(\frac{d_{\epsilon,N}}{\mu_\epsilon})\|_\infty + M(\epsilon)\|\partial_{00}(\frac{d_{\epsilon,N}}{\mu_\epsilon})\|_\infty + M(\epsilon)\|\partial_0 a_0\|_\infty + \|a_{\alpha,\beta}\|_\infty + \|Da_{\alpha,\beta}\|_\infty \\ + \|\langle y\rangle b_\alpha\|_\infty + \|\langle y\rangle^2 c\|_\infty < \delta, \end{aligned}$$

and

$$\delta^{-1} < \frac{d_{\epsilon,N}}{\mu_\epsilon}(\rho y_0) < M(\epsilon)\delta \ \ \text{for all } y_0 \in \mathbb{R}$$

for some positive constant δ, then there exists a unique solution ϕ to equation (5.1) satisfying

$$\int_{\mathcal{D}_1} \phi(y_0, y) Z_j(y) dy = 0, \ \forall y_0 \in \mathbb{R}, \ j = 0, 1, \cdots, N+1.$$

Furthermore, one has

$$|\phi(y_0, y)| \lesssim R^{\tau-\sigma} \langle y \rangle^{-\tau} \|h\|_{2+\sigma}. \tag{5.6}$$

Remark 5.1. *(1) In the intermediate region $|y| \sim R$, by Proposition 5.2, it follows that*

$$\|\phi\|_\sigma \lesssim \|h\|_{2+\sigma}.$$

(2) The estimate (5.6) is deteriorated in the interior. However, when we apply the linear theory to solve the reduced inner problem (4.24), taking τ close to 2 will be sufficient for our purpose.

6. The inner problem

In this section we drop all the tildes in the projected inner problem (4.24) for simplicity. We solve problem (4.24) in a $2l/\rho$-period (in y_0) manner as follows

$$\begin{cases} \mathcal{L}(\phi) = S_\epsilon(\mathbf{w}) + \mathbf{N}(\phi) + \sum\limits_{j=0}^{N+1} c_j(\rho y_0) Z_j(y), & \text{in } \mathbb{S}_\rho \times D_R \\ \phi(y_0, y) = \phi(y_0 + \frac{2l}{\rho}, y), & \text{for all } (y_0, y) \in \mathbb{S}_\rho \times D_R \\ \phi = 0, & \text{on } \partial(\mathbb{S}_\rho \times D_R) \end{cases}$$

where $\mathcal{L}(\phi) = \mathcal{A}\phi + p w^{p-1}\phi$ and

$$\mathbf{N}(\phi) = p(w^{p-1} - \mathbf{w}^{p-1})\phi - \mathtt{N}(\zeta_{2\delta}^\epsilon \phi + \psi(\phi)) + (\zeta_{2\delta}^\epsilon)^{p-1} p \mathbf{w}^{p-1} \psi(\phi) \tag{6.1}$$

with

$$\mathtt{N}(\phi) = \mu_\epsilon^{-\frac{(N-2)\epsilon}{2}} (\mathbf{w} + \phi)^{p-\epsilon} - \mu_\epsilon^{-\frac{(N-2)\epsilon}{2}} \mathbf{w}^{p-\epsilon} - p \mathbf{w}^{p-1}\phi.$$

Recall from (3.30) that

$$S_\epsilon(\mathbf{w}) = S_\epsilon(\mathtt{W}) = \epsilon^2 \left(\rho^2 a_0 e''(\rho y_0) + |\lambda_0| e(\rho y_0) \right) \chi_\epsilon Z_0 + \mathtt{E},$$

where $\mathtt{E} := \epsilon^2 S_1 + N_1(\Pi) + \sum\limits_{i=0}^{N+1} \alpha_i Z_i$ with $N_1(\Pi)$ defined in (3.31).

We consider the inner problem

$$\begin{cases} a_0 \partial_0^2 \phi + \Delta_y \phi + \tilde{\mathcal{A}}\phi + p w^{p-1}\phi = H(\phi, \psi, \mu, d, e) + \sum\limits_{j=0}^{N+1} c_j(\rho y_0) Z_j(y), \ \text{in } \mathbb{S}_\rho \times D_R \\ \phi(y_0, y) = \phi(y_0 + \frac{2l}{\rho}, y), \ \text{for all } (y_0, y) \in \mathbb{S}_\rho \times D_R \\ \phi(y_0, y) = 0, \ \text{on } \partial(\mathbb{S}_\rho \times D_R) \\ \int_{D_R} \phi(y_0, y) Z_i(y) dy = 0, \ \text{for all } i = 0, 1, \cdots, N+1 \text{ and all } y_0 \in \mathbb{S}_\rho, \end{cases} \tag{6.2}$$

where

$$H(\phi, \psi, \mu, d, e) := S_\epsilon(\mathtt{W}) + \mathbf{N}(\phi).$$

Our aim is to find ϕ by using the linear theory developed in Section 5. Here μ, d and e satisfy (3.24). Note that the Lipschitz dependence of $\mathtt{E}$ on μ, d and e is

$$\|\mathtt{E}(\mu_1, d_1, e_1) - \mathtt{E}(\mu_2, d_2, e_2)\|_\infty \lesssim \epsilon^2 \|(\mu_1 - \mu_2, d_1 - d_2, e_1 - e_2)\|.$$

Consider the fixed point problem

$$\phi = T(S_\epsilon(\mathtt{W}) + \mathbf{N}(\phi)) := \mathtt{A}(\phi). \tag{6.3}$$

We solve the above problem in the function space

$$\mathcal{M} = \{\phi : \|\phi\|_* \le \Lambda_1 \epsilon^{2(1-\vartheta)}\},$$

where the norm $\|\cdot\|_*$ is defined as in (4.6) and $\vartheta > 0$ is close to 0. The first fact we show is that the mapping $\mathtt{A}$ maps $\mathcal{M}$ to itself. We estimate term by term as follows.

From (3.30) and (3.32), we observe that $S_\epsilon(\mathtt{W})$ is independent of ϕ and it satisfies

$$\|S_\epsilon(\mathtt{W})\|_{2+\sigma} \lesssim \epsilon^{2(1-\vartheta)}. \tag{6.4}$$

The nonlinear term satisfies

$$\begin{aligned}\|\mathbf{N}(\phi)\|_{2+\sigma} \lesssim\ & \|(w^{p-1} - \mathbf{w}^{p-1})\phi\|_{2+\sigma} + \|\eta^\epsilon_{\delta,2R}\mathtt{N}(\eta^\epsilon_{\delta,2R}\phi + \psi(\phi))\|_{2+\sigma} \\ &+ \|(\eta^\epsilon_{\delta,2R})^{p-1}\mathbf{w}^{p-1}\psi(\phi)\|_{2+\sigma}.\end{aligned}$$

By (4.6), we estimate

$$\begin{aligned}\|(w^{p-1} - \mathbf{w}^{p-1})\phi\|_{2+\sigma} &\lesssim \| \left((w + \epsilon e Z_0 + \Pi)^{p-1} - w^{p-1}\right)\phi\|_{2+\sigma} \\ &\lesssim \|w^{p-2}(\epsilon e Z_0 + \Pi)\phi\|_{2+\sigma} \\ &\lesssim \sup_{\mathbb{S}_\rho \times D_R} \langle y\rangle^{2+\sigma}\langle y\rangle^{N-6}(\epsilon e Z_0 + \epsilon\langle y\rangle^{-\sigma})|\phi| \\ &\lesssim \epsilon R^{\tau-\sigma}\|\phi\|_*.\end{aligned} \tag{6.5}$$

Since $\|\psi\|_\infty \lesssim R^{-\sigma}\|\phi\|_*$, we get that for $(y_0, y) \in \mathbb{S}_\rho \times D_R$

$$|\eta^\epsilon_{\delta,2R}\phi + \psi| \lesssim \left(R^{-\sigma} + R^{\tau-\sigma}\langle y\rangle^{-\tau}\right)\|\phi\|_* \lesssim R^{\tau-\sigma}\langle y\rangle^{-\tau}\|\phi\|_*. \tag{6.6}$$

When $N = 5, 6$ we know that $w \gtrsim |\eta^\epsilon_{\delta,2R}\phi + \psi|$ if

$$|y| \lesssim R^{\frac{\sigma-\tau}{N-2-\tau}}\|\phi\|_*^{-\frac{1}{N-\tau-2}},$$

and vice versa. Recall that $R = R(\epsilon) = \epsilon^{-\theta_*}$ with $0 < \theta_* < 1$. We denote

$$R_1 := R^{\frac{\sigma-\tau}{N-2-\tau}}\|\phi\|_*^{-\frac{1}{N-\tau-2}} \sim \epsilon^{\frac{\theta_*(\tau-\sigma)}{N-2-\tau} - \frac{2(1-\vartheta)}{N-\tau-2}}. \tag{6.7}$$

Therefore, we can perform the Taylor expansion as follows. Using (6.6), we obtain that

$$\begin{aligned}\|\eta^\epsilon_{\delta,2R}\mathtt{N}(\eta^\epsilon_{\delta,2R}\phi + \psi(\phi))\|_{2+\sigma} &\lesssim \sup_{|y|\lesssim R_1} |\langle y\rangle^{2+\sigma}w^{p-2}(\phi+\psi)^2| \\ &\quad + \sup_{R_1 \lesssim |y| \lesssim R} \langle y\rangle^{2+\sigma}\left(R^{\tau-\sigma}\langle y\rangle^{-\tau}\|\phi\|_*\right)^p \\ &\lesssim R^{2(\tau-\sigma)}\|\phi\|_*^2 + R_1^{2+\sigma-p\tau}R^{p(\tau-\sigma)}\|\phi\|_*^p.\end{aligned} \tag{6.8}$$

For the last term in the nonlinear term, since $|\psi| \lesssim R^{-\sigma}\|\phi\|_*$, we get

$$\|(\eta^\epsilon_{\delta,2R})^{p-1}\mathbf{w}^{p-1}\psi(\phi)\|_{2+\sigma} \lesssim R^{-\sigma}\sup_{|y|\lesssim R}\langle y\rangle^{\sigma-2}\|\phi\|_* = R^{-\sigma}\|\phi\|_* \tag{6.9}$$

Collecting all the terms above from (6.5), (6.8) and (6.9), we obtain that for $R(\epsilon) = \epsilon^{-\theta_*}$ with θ_* small

$$\|\mathbf{N}(\phi)\|_{2+\sigma} \lesssim \epsilon R^{\tau-\sigma}\|\phi\|_* + R^{2(\tau-\sigma)}\|\phi\|_*^2 + R_1^{2+\sigma-p\tau}R^{p(\tau-\sigma)}\|\phi\|_*^p + R^{-\sigma}\|\phi\|_*. \tag{6.10}$$

Therefore, by (6.4) , (6.10), (6.3) and Proposition 5.2, we have

$$\|\mathtt{A}(\phi)\|_* \lesssim \|H\|_{2+\sigma} \lesssim \epsilon^2 + \epsilon R^{\tau-\sigma}\|\phi\|_* + R^{2(\tau-\sigma)}\|\phi\|_*^2 + R_1^{2+\sigma-p\tau}R^{p(\tau-\sigma)}\|\phi\|_*^p + R^{-\sigma}\|\phi\|_*. \tag{6.11}$$

We observe from (6.11) that, in order to make $\mathtt{A}$ map $\mathcal{M}$ to itself, the following relations should be satisfied

$$\begin{cases} \epsilon R^{\tau-\sigma} \le C \\ R^{2(\tau-\sigma)}\|\phi\|_* \le C \\ R_1^{2+\sigma-p\tau}R^{p(\tau-\sigma)}\|\phi\|_*^{p-1} \le C \end{cases}$$

for some constant C when ϵ is small. Recall that $R = R(\epsilon) = \epsilon^{-\theta_*}$. By (6.7) and $\|\phi\|_* \sim \epsilon^{2(1-\vartheta)}$, we obtain that

$$\begin{cases} 1-\theta_*(\tau-\sigma) > 0 \\ 2(1-\vartheta) - 2\theta_*(\tau-\sigma) > 0 \\ \frac{\theta_*(\tau-\sigma)-2(1-\vartheta)}{N-2-\tau}(2+\sigma-p\tau) - \theta_* p(\tau-\sigma) + 2(1-\vartheta)(p-1) > 0. \end{cases} \tag{6.12}$$

In our setting, τ is chosen slightly larger than 2 and $\vartheta > 0$ is close to 0. When $N = 5$, $\sigma \approx 1$. When $N = 6$, $\sigma \approx 2$. Elementary computations show that (6.12) is satisfied if $\theta_* < 1$. Thus, by $\theta_* < 1$ and (6.11), it follows that for ϵ small

$$\|\mathtt{A}(\phi)\|_* \le C\|\phi\|_*$$

where C is some positive constant. Therefore, we conclude that for $\phi \in \mathcal{M}$ with Λ_1 fixed sufficiently large, $\mathtt{A}(\phi) \in \mathcal{M}$.

It remains to show that $\mathtt{A}$ is a contraction mapping in $\mathcal{M}$. From (6.3), (6.10) and (6.11), we see that for $\phi^{(1)}, \phi^{(2)} \in \mathcal{M}$

$$\begin{aligned}\|\mathtt{A}(\phi^{(1)}) - \mathtt{A}(\phi^{(2)})\|_* &\lesssim \epsilon R^{\tau-\sigma}\|\phi^{(1)}-\phi^{(2)}\|_* + \epsilon^2 R^{2(\tau-\sigma)}\|\phi^{(1)}-\phi^{(2)}\|_* \\ &\quad + \epsilon^{2(p-1)}R_1^{2+\sigma-p\tau}R^{p(\tau-\sigma)}\|\phi^{(1)}-\phi^{(2)}\|_* + R^{-\sigma}\|\phi^{(1)}-\phi^{(2)}\|_*.\end{aligned}$$

By our choices of θ_*, τ, ϑ and σ in the system (6.12), we already see that

$$\|\mathtt{A}(\phi^{(1)}) - \mathtt{A}(\phi^{(2)})\|_* \le o(1)\|\phi^{(1)}-\phi^{(2)}\|_*,$$

where $o(1) \to 0$ as $\epsilon \to 0^+$. It then follows that $\mathtt{A}$ is a contraction mapping in the function space $\mathcal{M}$ for ϵ sufficiently small. Therefore, the Contraction Mapping Theorem implies the existence of the solution ϕ.

Moreover, by a similar argument as in [18], we find that the Lipschitz dependence of T on the parameters is given by

$$\|T_{(\mu_1,d_1,e_1)} - T_{(\mu_2,d_2,e_2)}\| \lesssim R^{-\sigma}\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|,$$

and for $\|\phi\|_* \lesssim \epsilon^{2(1-\vartheta)}$, we have

$$\|\mathbf{N}_{(\mu_1,d_1,e_1)}(\phi) - \mathbf{N}_{(\mu_2,d_2,e_2)}(\phi)\|_{2+\sigma} \lesssim R^{-\sigma}\epsilon^{2(1-\vartheta)}\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

By a fixed point argument, we see that

$$\|\phi_{(\mu_1,d_1,e_1)} - \phi_{(\mu_2,d_2,e_2)}\|_* \lesssim R^{-\sigma}\epsilon^{2(1-\vartheta)}\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

We have thus proved the existence of the following

Proposition 6.1. *For sufficiently small ϵ and suitable parameters μ, d and e satisfying*

$$\|(\mu, d, e)\| = \|\mu\|_a + \|d\|_d + \|e\|_e \lesssim 1,$$

the problem (6.2) has a unique solution $\phi = \phi(\mu, d, e)$ satisfying

$$\|\phi\|_* \lesssim \epsilon^{2(1-\vartheta)}.$$

Furthermore, ϕ depends Lipschitz continuously on μ, d and e with

$$\|\phi_{(\mu_1,d_1,e_1)} - \phi_{(\mu_2,d_2,e_2)}\|_* \lesssim R^{-\sigma}\epsilon^{2(1-\vartheta)}\|(\mu_1-\mu_2, d_1-d_2, e_1-e_2)\|.$$

7. Choice of the parameter functions μ, d and e

In this section, we shall choose the parameter functions μ, d and e such that

$$c_j(\rho y_0) = 0,\ j = 0, 1, \cdots, N+1. \tag{7.1}$$

are satisfied. Multiplying equation (6.2) with Z_j and integrating in y over D_R imply that the reduced system (7.1) is equivalent to

$$\int_{D_R}\Big(a_0\partial_0^2\phi + \Delta_y\phi + \tilde{A}\phi + pw^{p-1}\phi - H\Big)Z_j dy = 0 \tag{7.2}$$

for all $y_0 \in \mathbb{S}_\rho$ and $j = 0, 1, \cdots, N+1$. Recall from problem (6.2) that

$$H = S_\epsilon(\mathtt{W}) + \mathbf{N}(\phi),$$

where $S_\epsilon(\mathtt{W})$ and $\mathbf{N}(\phi)$ are defined in (3.30) and (6.1), respectively. Since $S_\epsilon(\mathtt{W})$ and $\mathbf{N}(\phi)$ involve μ, d and e, our full problem is reduced to a system involving these parameter functions. Recall that

$$\mathbf{N}(\phi) = p(w^{p-1} - \mathbf{w}^{p-1})\phi - \mathtt{N}(\eta^{\epsilon}_{\delta,2R}\phi + \psi(\phi)) + (\eta^{\epsilon}_{\delta,2R})^{p-1}p\mathbf{w}^{p-1}\psi(\phi)$$

where

$$\mathtt{N}(\phi) = \mu_\epsilon^{\frac{(N-2)\epsilon}{2}}(\mathbf{w}+\phi)^{p-\epsilon} - \mu_\epsilon^{\frac{(N-2)\epsilon}{2}}\mathbf{w}^{p-\epsilon} - p\mathbf{w}^{p-1}\phi,$$

and

$$S_\epsilon(\mathtt{W}) = \epsilon^2 S_1 + \epsilon^2\left(\rho^2 a_0 e''(\rho y_0) + |\lambda_0| e(\rho y_0)\right)\chi_\epsilon Z_0 + N_1(\Pi) + \sum_{i=0}^{N+1}\alpha_i Z_i,$$

where $N_1(\Pi)$ is the nonlinear term defined as

$$N_1(\Pi) = \mu_\epsilon^{\frac{N-2}{2}\epsilon}[(\mathtt{w}+\Pi)^{p-\epsilon} - \mathtt{w}^{p-\epsilon}] - pw^{p-1}\Pi. \tag{7.3}$$

Next we expand (7.2) in terms of μ, d, e.

7.1. Projections of $S_\epsilon(\mathtt{W})$ on Z_j, $j = 0, 1, \cdots, N+1$.

For $j = 0, 1, \cdots, N+1$, one has

$$\begin{aligned}\int_{D_R} S_\epsilon(\mathtt{W})Z_j &= \epsilon^2\int_{D_R} S_1 Z_j + \int_{D_R}\epsilon^2\left(\rho^2 a_0 e''(\rho y_0) + \lambda_1 e(\rho y_0)\right)\chi_\epsilon Z_0 Z_j \\ &\quad + \int_{D_R} N_1(\Pi)Z_j + \alpha_j\int_{D_R} Z_j^2 \\ &= \int_{D_R} S_\epsilon(\mathtt{w})Z_j + \int_{D_R} N_1(\Pi)Z_j + \alpha_j\int_{D_R} Z_j^2,\end{aligned} \tag{7.4}$$

where $S_\epsilon(\mathtt{w})$ is defined in (3.26).

It turns out that the size of projections on Z_0, Z_N and Z_{N+1} are much larger than that of Z_j directions with $j = 1, \cdots, N-1$. We proceed to compute the projections of $S_\epsilon(\mathtt{w})$ and $N_1(\Pi)$ along different directions as follows. Here we mainly follow the results in [18, Section 5].

7.2. **Projections of $S_\epsilon(\mathtt{w})$ on Z_j, $j = 0, \cdots, N+1$.** We use the expansion of $S_\epsilon(\mathtt{w})$ (3.25) to compute the projection of $S_\epsilon(\mathtt{w})$ onto Z_i, $i = 0, 1, \cdots, N+1$. Suppose the parameter functions μ, d and e satisfy

$$\|(\mu, d, e)\| := \|\mu\|_a + \|d\|_d + \|e\|_e \le c,$$

where the norms are defined in (3.5), (3.6), (3.7), (3.8) and (3.23). With suitable choices of μ_ϵ^0 and $d_{\epsilon,N}$ as in (3.4) and e_0 as in (3.22), the main order terms in the projections of $S_\epsilon(\mathtt{w})$ on Z_j are eliminated, and thus we obtain

$$\begin{aligned}\int_{D_R} S_\epsilon(\mathtt{w}) Z_k = \epsilon^{2+\frac{1}{N-2}} \Big(\int_{\mathbb{R}^N} Z_k^2 \Big) \big[\mu_0(-d_k'' + R_{0j0k} d_j) + \alpha_k(\rho y_0) \\ + \epsilon \beta_k(\rho y_0; \mu, d, e) \big] + \epsilon^3 r\end{aligned} \tag{7.5}$$

$$\begin{aligned}\varpi \int_{D_R} S_\epsilon(\mathtt{w}) Z_N = \epsilon^{2+\frac{1}{N-2}} [B\bar{h}_{00}\mu + C\bar{h}_{00} d_N + \alpha_N(\rho y_0) + \epsilon \beta_N(\rho y_0; \mu, d, e)] \\ - \epsilon^{3+\frac{1}{N-2}} \left(\int_{\mathbb{R}^N} Z_N^2 \right) \mu_0 d_N'' + \epsilon^4 r\end{aligned} \tag{7.6}$$

$$\begin{aligned}\int_{D_R} S_\epsilon(\mathtt{w}) Z_{N+1} = \epsilon^2 [A\bar{h}_{00}\mu + B\bar{h}_{00} d_N + \alpha_{N+1}(\rho y_0) + \epsilon \beta_{N+1}(\rho y_0; \mu, d, e)] \\ - \epsilon^{3+\frac{2}{N-2}} \left(\int_{\mathbb{R}^N} Z_{N+1}^2 \right) \mu_0 \mu'' + \epsilon^4 r\end{aligned} \tag{7.7}$$

$$\begin{aligned}\int_{D_R} S_\epsilon(\mathtt{w}) Z_0 = \epsilon^2 \left(\int_{\mathbb{R}^N} Z_0^2 \right) \Bigg\{ \rho^2 a_0 e'' + |\lambda_0| e + \alpha_0(\rho y_0) \\ - 2(\mathrm{Tr}_{\bar{g}} \bar{h} - \bar{h}_{00}) \left(\int_{\mathbb{R}^N} \partial_{ii} w Z_0 \right) d_N + \sum_i \Big[(d_i')^2 \\ - \frac{1}{3} R_{ikil} d_k d_l + a_{Nk}^{ii} d_k d_{0,N} + 4\bar{h}_{0j} d_j d_{0,N} \Big] \left(\int_{\mathbb{R}^N} \partial_{ii} w Z_0 \right) \\ + \epsilon^2 \beta_0(\rho y_0; \mu, d, e) \Bigg\} + \epsilon^4 r.\end{aligned} \tag{7.8}$$

In the above expressions (7.5)-(7.8), R_{ijkl} is the component of the curvature tensor defined in (2.1), α_k and β_k are smooth and uniformly bounded in ϵ. Note that α_k and β_k does not depend on μ', d' and e'. Function r is of the following form

$$h_0(\rho y_0) \left[h_1(\mu, d, e, \mu', d', e') + o(1) h_2(\mu, d, e, \mu', d', e', \mu'', d'', e'') \right],$$

where h_0, h_1 and h_2 are smooth and uniformly bounded in ϵ. A, B and C are constants depending only on the dimension with $AC - B^2 > 0$. ϖ is a constant depending on the dimension and the smooth functions

$$\mu_0, d_{0,N}, e_0, \mu_1, d_{1,N}, e_1 : (-l, l) \to \mathbb{R}$$

are defined in (3.4), (3.20) and (3.22).

Since the projections (7.5)-(7.8) are exactly the same as that of [18], we omit the proof here. A proof can be found in [18, Appendix].

7.3. **Projections of $N_1(\Pi)$ and $\sum \alpha_i Z_i$.** By the computations in Section 3.5 about the estimates of Π, we have the following. For ϵ sufficiently small and $N = 5, 6$, it holds that

$$\int_{D_R} N_1(\Pi) Z_j dy = \epsilon^{2(1-\vartheta)} h_0(\rho y_0), \quad \text{for } j = 0, 1, \cdots, N+1, \tag{7.9}$$

where $N_1(\Pi)$ is defined in (7.3), $h_0(\rho y_0)$ is a smooth function of ρy_0. Indeed, by (3.32) and the decay of Z_j, $j = 0, 1, \cdots, N+1$, we can easily get the desired estimate.

Note that in (7.9), $h_0(\rho y_0)$ is independent of μ, d and e. From a quite similar argument as in [18, Appendix], we can eliminate the largest term $\epsilon^{2(1-\vartheta)} h_0(\rho y_0)$ by solving a system like [18, Appendix (9.17)]. Therefore, the new projection becomes

$$\int_{D_R} N_1(\Pi) Z_j dy = o(1)\epsilon^3, \quad \text{for } j = 0, 1, \cdots, N+1. \tag{7.10}$$

On the other hand, by (3.29) and the fact that $\vartheta > 0$ is close to 0, we see that

$$\alpha_j \int_{D_R} Z_j^2 = o(1)\epsilon^{3-\vartheta}, \tag{7.11}$$

which is of smaller order compared with the projections of $S_\epsilon(\mathtt{w})$'s for ϵ small.

7.4. **Projections of $\mathbf{N}(\phi)$.** From (6.10), $\|\phi\|_* \sim \epsilon^{2(1-\vartheta)}$ and the definition of $R = R(\epsilon)$ as in (4.1), one has

$$\begin{aligned}\|\mathbf{N}(\phi)\|_{2+\sigma} \lesssim\ & \epsilon^{3-2\vartheta-\theta_*(\tau-\sigma)} + \epsilon^{4(1-\vartheta)-2\theta_*(\tau-\sigma)} \\ & + \epsilon^{\frac{\theta_*(\tau-\sigma)-2(1-\vartheta)}{N-2-\tau}(2+\sigma-p\tau)-\theta_* p(\tau-\sigma)+2(1-\vartheta)p} + \epsilon^{2(1-\vartheta)+\sigma\theta_*}.\end{aligned}$$

Now we choose θ_* such that the projections of $\mathbf{N}(\phi)$ are of smaller order compared with the leading order of $S_\epsilon(\mathtt{w})$'s for R sufficiently large (namely ϵ sufficiently small). More precisely, by (7.5), the projection of $S_\epsilon(\mathtt{w})$ along Z_j $(j = 1, \cdots, N-1, N)$ is of order $\epsilon^{2+\frac{1}{N-2}}$. Thus, θ_* satisfies the following inequalities

$$\begin{cases} 3 - 2\vartheta - \theta_*(\tau-\sigma) > 2 + \frac{1}{N-2} \\ 4(1-\vartheta) - 2\theta_*(\tau-\sigma) > 2 + \frac{1}{N-2} \\ \frac{\theta_*(\tau-\sigma)-2(1-\vartheta)}{N-2-\tau}(2+\sigma-p\tau) - \theta_* p(\tau-\sigma) + 2(1-\vartheta)p > 2 + \frac{1}{N-2} \\ 2(1-\vartheta) + \sigma\theta_* > 2 + \frac{1}{N-2} \end{cases} \tag{7.12}$$

We know that $\tau \approx 2$, $\sigma \approx N-4$ and $\vartheta \approx 0$. To make the projection of $\mathbf{N}(\phi)$ along Z_j comparatively smaller than $S_\epsilon(\mathtt{w})$'s, a sound choice of θ_* satisfying system (7.12) is

$$\theta_* = \frac{1+\nu}{(N-2)\sigma} \text{ with } R(\epsilon) = \epsilon^{-\theta_*} \text{ and } \nu > 0 \text{ small.} \tag{7.13}$$

We can easily check the other directions Z_0 and Z_{N+1} in a similar way.

In conclusion, with such θ_* in (7.13), we obtain that for $i = 0, 1, \cdots, N, N+1$

$$\int_{D_R} \mathbf{N}(\phi) Z_i dy = o(1) \int_{D_R} S_\epsilon(\mathtt{w}) Z_i dy. \tag{7.14}$$

7.5. **Projections of $\mathcal{L}(\phi)$.** Recall that $\mathcal{L}(\phi) = \mathcal{A}\phi + pw^{p-1}\phi$ and $\mathcal{A}\phi = a_0\partial_0^2\phi + \Delta_y\phi + \tilde{\mathcal{A}}\phi$. Since the differential operator $\tilde{\mathcal{A}}$ is a small perturbation of Δ_y of order ϵ, we have

$$|\tilde{\mathcal{A}}\phi| \lesssim \epsilon|\phi| \lesssim \epsilon R^{\tau-\sigma}\|\phi\|_*,$$

where we have used the definition of the norm $\|\cdot\|_*$ in (4.6). By $\|\phi\|_* \sim \epsilon^{2(1-\vartheta)}$ and the choice of θ_* in (7.13), we obtain

$$|\tilde{\mathcal{A}}\phi| \lesssim \epsilon^{3-2\vartheta-\theta_*(\tau-\sigma)} = o(1)\epsilon^{2+\frac{1}{N-2}} \tag{7.15}$$

for ϵ small. For the remaining terms, since the solution ϕ we get is orthogonal to Z_j in D_R, we have

$$\begin{aligned}
&\int_{D_R}\left(a_0\partial_0^2\phi + \Delta_y\phi + pw^{p-1}\phi\right)Z_j dy = \int_{D_R}\left(\Delta_y\phi + pw^{p-1}\phi\right)Z_j dy\\
&= \int_{D_R}\left(\Delta_y Z_j + pw^{p-1}Z_j\right)\phi dy + \int_{\partial D_R} Z_j\partial_\nu\phi dS - \int_{\partial D_R}\phi\partial_\nu Z_j dS\\
&\lesssim R^{-\sigma}\|\phi\|_* = o(1)\epsilon^{2+\frac{1}{N-2}}
\end{aligned} \tag{7.16}$$

for ϵ small, where we have used the integration-by-parts formula and (7.13). In conclusion, combining (7.15) and (7.16), we have

$$\int_{D_R}\mathcal{L}(\phi)Z_j dy = o(1)\epsilon^{2+\frac{1}{N-2}} \tag{7.17}$$

for $j = 0, 1, \cdots, N+1$.

7.6. **Reduced equations for μ, d, e.** By the computations above, the reduced system (7.2) are equivalent to a system of ODEs for μ, d, e. We assume that

$$\|(\mu, d, e)\| := \|\mu\|_a + \|d\|_d + \|e\|_e \le c. \tag{7.18}$$

By collecting (7.4), (7.5), (7.6), (7.7), (7.8), (7.10), (7.11), (7.14) and (7.17), we know that the reduced system (7.2) are achieved if (e, d, μ) satisfies the following system of ODEs

$$\begin{cases}
\mathcal{L}_0(e) := \rho^2 a_0 e'' + |\lambda_0|e + \gamma_0 d_N = -\alpha_0(\rho y_0) - Q_0(d) + \epsilon^2 M_0(\rho y_0; \mu, d, e)\\
\mathcal{L}_k(d_k) := -d_k'' + R_{0j0k}d_j = -\alpha_k(\rho y_0) + \epsilon M_k(\rho y_0; \mu, d, e), \quad k = 1, \cdots, N-1,\\
\mathcal{L}_N(d_N) := -\epsilon C_N\varpi\mu_0 d_N'' + B\bar{h}_{00}\mu + C\bar{h}_{00}d_N = -\alpha_N(\rho y_0) + \epsilon M_N(\rho y_0; \mu, d, e)\\
\mathcal{L}_{N+1}(\mu) := -\epsilon^{\frac{N}{N-2}}C_{N+1}\mu_0\mu'' + A\bar{h}_{00}\mu + B\bar{h}_{00}d_N\\
\qquad\qquad = -\alpha_{N+1}(\rho y_0) + \epsilon M_{N+1}(\rho y_0; \mu, d, e),
\end{cases} \tag{7.19}$$

where

$$\begin{aligned}
&C_N := \int_{\mathbb{R}^N} Z_N^2,\ C_{N+1} := \int_{\mathbb{R}^N} Z_{N+1}^2,\\
&\gamma_0 := -2(\mathrm{Tr}_{\bar{g}}\bar{h} - \bar{h}_{00})\left(\int_{\mathbb{R}^N}\partial_{ii}wZ_0\right),
\end{aligned} \tag{7.20}$$

and

$$Q_0(d) = \sum_i\left[(d_i')^2 - \frac{1}{3}R_{ikil}d_k d_l + a_{Nk}^{ii}d_k d_{0,N} + 4\bar{h}_{0j}d_j d_{0,N}\right]\left(\int_{\mathbb{R}^N}\partial_{ii}wZ_0\right).$$

For $j = 0, 1, \cdots, N, N+1$, the operator $M_j(\rho y_0; \mu, d, e)$ can be decomposed into the following form

$$M_j(\rho y_0; \mu, d, e) = A_j(\rho y_0; \mu, d, e) + K_j(\rho y_0; \mu, d, e)$$

where K_j is uniformly bounded in $L^\infty(-l,l)$ for (μ,d,e) satisfying (7.18) and is compact, A_j depends on $(\mu,d,e,\mu',d',e',\mu'',d'',e'')$ and satisfies

$$\|A_j(\mu_1,d_1,e_1)-A_j(\mu_2,d_2,e_2)\|_\infty \lesssim o(1)\|(\mu_1,d_1,e_1)-(\mu_2,d_2,e_2)\|,$$

in which the dependence of A_j on μ'', d'' and e'' is linear.

7.7. **Linear theory for the ODE system** (7.19)**.** For $j=0,1,\cdots,N,N+1$, we first develop a linear theory concerning the invertibility of $\mathcal{L}_j$ in a L^∞ manner.

We seek $2l$-periodic solutions of the following problem

$$\mathcal{L}_{N+1}(\mu)=h_1,\quad \mathcal{L}_N(d)=h_2, \tag{7.21}$$

where$\|h_1\|_\infty+\|h_2\|_\infty<+\infty$. We have the following existence and a priori estimates for the problem (7.21).

Lemma 7.1. *Assume that $A>0$, $C>0$ and $AC-B^2>0$. If $\|h_1\|_\infty+\|h_2\|_\infty<+\infty$, then there exists a $2l$-periodic solution (μ,d) of (7.21) such that*

$$\|\mu\|_\infty+\|d\|_\infty+\epsilon^{\frac{N}{2(N-2)}}\|\mu'\|_\infty+\epsilon^{\frac12}\|d'\|_\infty\lesssim\|h_1\|_\infty+\|h_2\|_\infty.$$

Proof. The associated energy functional for the operators $\mathcal{L}_N$ and $\mathcal{L}_{N+1}$ is given by

$$\begin{aligned}F(\mu,d)=\int_{-l}^{l}\Big[&\epsilon^{\frac{N}{N-2}}\mu_0(\mu')^2+\epsilon\mu_0(d')^2+\epsilon^{\frac{N}{N-2}}\mu_0'\mu\mu'+\epsilon\mu_0'dd'\\&+(A\mu^2+2Bd\mu+Cd^2)\bar{h}_{00}+h_1\mu+h_2d\Big]dx_0.\end{aligned}$$

Since $A>0$, $C>0$ and $AC-B^2>0$, for $\epsilon>0$ small, we have

$$F(\mu,d)\ge c,$$

where c is a positive constant. Hence, the existence of solution to (7.21) follows.

The proof of the a priori estimate is the same as that in [18, Lemma 8.1]. □

Now we consider the invertibility of

$$\mathcal{L}_0(e):=\rho^2a_0e''+|\lambda_0|e+\gamma_0d=f. \tag{7.22}$$

We perform the Liouville transform as follows.

$$\mathfrak{m}=\int_{-l}^{l}\frac{1}{\sqrt{a_0(s)}}ds,\ t=\frac{\pi\int_{-l}^{s}\left(\sqrt{a_0(\theta)}\right)^{-1}d\theta}{\mathfrak{m}},$$

$$\tilde{\lambda}_0=\frac{\mathfrak{m}^2}{\pi^2}|\lambda_0|,\ y(t)=a_0^{-\frac14}(s)\tilde{e}(s),\ q(t)=\frac{\mathfrak{m}^2}{\pi^2}\left(a_0^{\frac14}\right)''a_0^{\frac34}.$$

After the Liouville transform, equation (7.22) for e gets reduced to

$$\begin{cases}\rho^2(y''+q(t)y)+\tilde{\lambda}_0y=\tilde{f}, & \text{in } (0,\pi)\\ y(0)=y(\pi),\ y'(0)=y'(\pi)\end{cases} \tag{7.23}$$

By directly applying the Sturm-Liouville theory to (7.23) together with the non-resonance condition

$$|k^2\epsilon^{2\frac{N-1}{N-2}}-\kappa^2|>\delta\epsilon^{\frac{N-1}{N-2}} \tag{7.24}$$

and

$$\kappa=\frac{\sqrt{|\lambda_0|}}{2\pi}\int_{-l}^{l}\frac{1}{\sqrt{a_0(s)}}ds, \tag{7.25}$$

we obtain the following existence and a priori estimates for e.

Lemma 7.2. *Assume that ϵ satisfies the non-resonance condition (7.24). If $f \in C(-l,l) \cap L^\infty(-l,l)$, then there exists a unique $2l$-periodic solution e of (7.22) satisfying*

$$\rho^2\|e''\|_\infty + \rho\|e'\|_\infty + \|e\|_\infty \lesssim \rho^{-1}\|f\|_\infty.$$

Furthermore, if $f \in C^2(-l,l)$, then

$$\rho^2\|e''\|_\infty + \rho\|e'\|_\infty + \|e\|_\infty \lesssim \|f''\|_\infty + \|f'\|_\infty + \|f\|_\infty.$$

Proof. See [18, Lemma 8.2]. □

7.8. **Final argument.**

Proof of Theorem 1.1. From the nondegenerate condition of the geodesic Γ (1.6), we have that for any $f \in L^\infty(-l,l)$, $k = 1, \cdots, N-1$, there exists a $2l$-periodic function d_k such that $\mathcal{L}_k(d_k) = f$ with

$$\|d_k''\|_\infty + \|d_k'\|_\infty + \|d_k\|_\infty \lesssim \|f\|_\infty. \tag{7.26}$$

Let $(\tilde\mu_0,\, \tilde d_{0,N},\, \tilde d_{0,k})$ be a solution to

$$\begin{cases} \mathcal{L}_k(\tilde d_{0,k}) = \alpha_k, \quad k = 1, \cdots, N-1, \\ \mathcal{L}_N(\tilde d_{0,N}) = \alpha_N \\ \mathcal{L}_{N+1}(\tilde\mu_0) = \alpha_{N+1}. \end{cases}$$

By Lemma 7.1 and (7.26), we obtain that

$$\epsilon\|\tilde d_{0,N}''\|_\infty + \epsilon^{\frac12}\|\tilde d_{0,N}'\|_\infty + \|\tilde d_{0,N}\|_\infty \le c, \tag{7.27}$$

$$\|\tilde d_{0,k}''\|_\infty + \|\tilde d_{0,k}'\|_\infty + \|\tilde d_{0,k}\|_\infty \le c, \tag{7.28}$$

and

$$\epsilon^{\frac{N}{N-2}}\|\tilde\mu_0''\|_\infty + \epsilon^{\frac{N}{2(N-2)}}\|\tilde\mu_0'\|_\infty + \|\tilde\mu_0\|_\infty \le c.$$

Now we consider

$$\mathcal{L}_0(\hat e_0) = -\gamma_0\tilde d_{0,N} - \alpha_0 - Q_0(\tilde d_0)$$

where $\tilde d_0 = (\tilde d_{0,1}, \cdots, \tilde d_{0,N})$ and γ_0 is defined in (7.20). Since α_0 and $Q_0(\tilde d_0)$ are regular, by (7.27), (7.28) and Lemma 7.2, we have that

$$\epsilon^{\frac{2N-2}{N-2}}\|\hat e_0''\|_\infty + \epsilon^{\frac{N-1}{N-2}}\|\hat e_0'\|_\infty + \|\hat e_0\|_\infty \le c. \tag{7.29}$$

Summarizing (7.27), (7.28) and (7.29), it holds that

$$\|(\tilde\mu_0, \tilde d_0, \hat e_0)\| \le c.$$

We assume that

$$\mu = \tilde\mu_0 + \tilde\mu_1, \quad d = \tilde d_0 + \tilde d_1, \quad e = \hat e_0 + \tilde e_1.$$

Then the original system (7.19) reduces to

$$\begin{cases} \mathcal{L}_0(\tilde e_1) = -\gamma_0\tilde d_{1,N} + \epsilon^2 M_0(\rho y_0; \mu, d, e) \\ \mathcal{L}_k(\tilde d_{1,k}) = \epsilon M_k(\rho y_0; \mu, d, e), \quad k = 1, \cdots, N-1, \\ \mathcal{L}_N(\tilde d_{1,N}) = \epsilon M_N(\rho y_0; \mu, d, e) \\ \mathcal{L}_{N+1}(\tilde\mu_1) = \epsilon M_{N+1}(\rho y_0; \mu, d, e). \end{cases} \tag{7.30}$$

A direct use of Schauder's fixed point theorem establishes the existence of $(\tilde{\mu}_1, \tilde{d}_1, \tilde{e}_1)$ solving system (7.30), whose proof can be found in [18]. We omit the details. □

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