to invoke Schauder's fixed point theorem. Specifically, $J(u, w) := (u', w')$ where $v = \Phi_w - u$ and $\mathcal{L}_{\widetilde{M}} u' = Q_v + w'$. Therefore, at a fixed point we have that

$$\mathcal{L}_{\widetilde{M}} v = \mathcal{L}_{\widetilde{M}} \Phi_w - \mathcal{L}_{\widetilde{M}} u = 1 - H_{\widetilde{M}} + w - Q_v - w = 1 - H_{\widetilde{M}} - Q_v.$$

We mention just a few of the differences from the surface case. First, the operator $\mathcal{L}_M$ no longer scales well under conformal change. Therefore, solving the linear problem requires some new techniques. We simplify how we solve the linear problem on the Delaunay pieces by realizing that we can project the linear operator onto the eigenspaces for $\mathbb{S}^{n-1}$ and thus treat the problem as an ODE on each eigenspace.

Going forward, there are a few questions one might consider. First, can we extend such a construction to more general manifolds? What requirements will be needed for Γ and what restrictions will need to be placed on the ambient manifold? Second, can one extend the hypersurface construction to include hypersurfaces with infinite topology? (This question is fairly straightforward and could be done by a graduate student.) Third, are the hypersurfaces that we construct non-degenerate? An affirmative answer would let us appeal to [4] to conclude that there exists a neighborhood in the moduli space that is a smooth manifold. One then might attempt to produce stronger fixed point estimates (contraction mappings) so that modifications to Γ result in surfaces that vary along a curve in the moduli space.

References

[1] C. Delaunay, *Sur la surface de revolution dont la courbure moyenne est constant*, J. Math. Pures Appl. **6** (1841), 309–320.
[2] H. Wente, *Counterexample to a conjecture of H. Hopf*, Pac. J. Math. **121** (1986), 193–243.
[3] M. Jleli, *End-to-end gluing of constant mean curvature hypersurfaces*, Ann. Fac. Sci. Toulouse Math. **18** (2009), 717–737.
[4] M. Jleli, *Moduli space theory of constant mean curvature hypersurfaces*, Adv. Nonlinear Stud. **9** (2009), 29–68.
[5] C. Breiner and N. Kapouleas, *Embedded constant mean curvature surfaces in Euclidean three-space*, Math. Ann. **360** (2014), 1041–1108.
[6] N. Kapouleas, *Complete constant mean curvature surfaces in Euclidean three-space*, Ann. Math. (2) **131** (1990), 239–330.

Finite total Q-curvature in conformal geometry and the CR geometry

Yi Wang
(joint work with Paul Yang)

1. Background

The study of non-compact complete surfaces with finite total Gaussian curvature dates back to the early 1930s. The works of Fiala, Huber, Cohn-Vossen demonstrate that the integral of the Gaussian curvature has rigid geometric consequences. The famous Fiala-Huber isoperimetric inequality states that if the integral of the positive part of Gaussian curvature is less than that on the half cylinder, then the isoperimetric inequality is valid on this complete surface. In higher dimensions, PDE aspect of Q-curvature has been intensively studied. However, the geometric implication of the Q-curvature has always been a mystery. It was conjectured by Bonk, Heinonen, Saksman that on conformally flat manifolds if the integral of the Q-curvature is less than that on the half cylinder, then the manifold is bi-Lipschitz to the Euclidean space. A few years ago, I proved this conjecture, with some non-uniform isoperimetric constant. Later, I improved this result by showing the full analog of Fiala-Huber type isoperimetric inequality on higher dimensional manifolds.

2. Isoperimetric inequality and Q-curvature in conformal geometry

The main result that I have presented is the following.

Theorem 1. *(W. '13) Suppose $(M^n, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ is a noncompact complete Riemannian manifold with normal metric. If its Q-curvature satisfies*

$$\alpha \overset{def}{=} \int_{M^n} Q_g^+ dv_g < c_n \tag{1}$$

and

$$\beta \overset{def}{=} \int_{M^n} Q_g^- dv_g < \infty, \tag{2}$$

then the manifold satisfies the isop inequality:

$$|\Omega|_g \leq C(\alpha, \beta, n)|\partial\Omega|_g^{n/(n-1)}. \tag{3}$$

In order to prove this theorem, I adopt some techniques from harmonic analysis. More precisely, the theory of A_p weights, especially the strong A_∞ weight is applied to solve the problem.

The Q-curvature is generally believed to have very rich geometric meanings. In a recently prepared preprint, I prove that under certain curvature conditions the integral of Q-curvature is quantized.

Theorem 2. *(W. '15) Suppose $(M^4, g) = (\mathbb{R}^4, e^{2u}|dx|^2)$ is a noncompact complete Riemannian manifold with normal metric. If M^4 embeds in $\mathbb{R}^5$ with*

$$\int_{M^4} |L|^4 dv_g < \infty, \tag{4}$$

with L being the second fundamental form, then

$$\int_{M^4} Q_g dv_g = 4\pi^2\mathbb{Z}.$$

Q-curvature is related to asymptotic behavior of the ends of local conformally flat manifolds. In the long run, it would be interesting to understand volume growth estimates, geodesic distance estimates, etc. using Q-curvature.

3. Q'-curvature in CR geometry

In CR geometry, if one considers conformally Heisenberg manifolds, it is known that the Q-curvature's integral is equal to zero. Therefore, the Q-curvature's integral is not the right quantity to study geometric properties. Recently, Case and Yang defined P_4' on general three dimensional CR manifolds. This is an operator which only acts on pluriharmonic functions. It satisfies the transformation law in a manner similar to that of the Q-curvature in the Riemannian setting, modular the space of pluriharmonic functions. Namely, for conformal change $\theta^u = e^u \theta_0$,

$$P_4'(f) = e^{2u}(P^u)_4'(f) \mod \mathcal{P}^+,$$

for all pluriharmonic functions f. The corresponding Q_4'-curvature satisfies

$$P_4'(u) + Q_4' = e^{2u}(Q^u)_4' \mod \mathcal{P}^+.$$

Theorem 3 (joint with Paul Yang, '15)**.** *Suppose the CR Q_4'-curvature of $(\mathbb{H}^1, e^u\theta)$ is nonnegative. The Webster scalar curvature is nonnegative at infinity, and u is a pluriharmonic funcion on $\mathbb{H}^1$. If*

$$\int_{\mathbb{H}^1} Q_4' e^{2u} dx < c_1, \tag{5}$$

then e^{2u} is an A_1 weight. Thus on such a conformal Heisenberg group, the isoperimetric inequality is valid. Moreover, the isoperimetric constant depends only on the integral of the Q_4'-curvature.

In our theorem, we provide further evidence that P_4' operator is the correct analogue on CR manifolds. First, analogous isoperimetric inequality holds when Q_4' is nonnegative. Also, nonnegativity of Webster scalar curvature at infinity implies the metric is normal.

References

[1] Y. Wang, *Isoperimetric inequality, Q-curvature and A_p weights*, Adv. Math. **281** (2015), 823-844
[2] Y. Wang, *Finite total Q-curvature on locally conformally flat manifolds*, preprint (2015)
[3] M. Muster, *Computing certain invariants of topological spaces of dimension three*, Topology **32** (1990), 100–120.
[4] Y. Wang, P. Yang, *On the isoperimetric inequality for conformally Heisenberg manifolds*, preprint (2015)

Scattering for a critical nonlinear wave equation in two space dimensions

Michael Struwe

abstract

In joint work with Martin Sack we show that the solutions to the Cauchy problem for a wave equation with critical exponential nonlinearity in 2 space dimensions scatter for arbitrary smooth, compactly supported initial data.