FRACTIONAL POINCARÉ INEQUALITIES IN VARIOUS SETTINGS
DISUGUAGLIANZE DI POINCARÉ FRAZIONARIE IN CONTESTI DIFFERENTI

YANNICK SIRE AND YI WANG

ABSTRACT. We survey on several fractional Poincaré inequalities in several contexts: the Euclidean case, the case of Lie groups and the case of Riemannian manifolds.

SUNTO. Presentiamo una breve rassegna su varie disuguaglianze di Poincaré frazionarie in contesti differenti: il caso dello spazio Euclideo, il caso dei gruppi di Lie, e il caso delle varietà Riemanniane.

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1. INTRODUCTION

The purpose of this article is to give a survey on various quite recent works on the derivation of fractional Poincare inequalities out of usual ones. By this, we mean a self-improving property from an $H^1 - L^2$ inequality to an $H^\alpha - L^2$ inequality for $\alpha \in (0,1)$. We will report on several works starting on the euclidean case endowed with a general measure, the case of Lie groups and Riemannian manifolds endowed also with a general measure and finally the case of conformally flat manifolds with finite total Q-curvature and a more transparent Gagliardo inequality.

2. THE CASE OF WEIGHTED POINCARÉ INEQUALITIES IN VARIOUS CONTEXTS

2.1. **The case of the Euclidean space.** Throughout this section, we denote by M a positive weight in $L^1(\mathbb{R}^n)$. We assume that M is a C^2 function and that this measure

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M satisfies the usual Poincaré inequality: there exists a constant $\lambda(M) > 0$ such that $\forall\, f \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} |\nabla f(y)|^2\, M(y)\, dy \geq \lambda(M) \int_{\mathbb{R}^n} \left| f(y) - \int_{\mathbb{R}^n} f(x)\, M(x)\, dx \right|^2 M(y)\, dy. \tag{1}$$

If the measure M can be written $M = e^{-V}$, this inequality is known to hold (see [BBCG08], or also [Vil09], Appendix A.19, Theorem 1.2, see also [DS90], Proof of Theorem 6.2.21 for related criteria) whenever there exist $a \in (0,1)$, $c > 0$ and $R > 0$ such that

$$\forall\, |x| \geq R, \quad a\, |\nabla V(x)|^2 - \Delta V \geq c. \tag{2}$$

In particular, the inequality (1) holds, for instance, when $M = (2\pi)^{-n/2} \exp\left(- |x|^2 /2\right)$ is the Gaussian measure, but also when $M(x) = e^{-|x|}$, and more generally when $M(x) = e^{-|x|^\alpha}$ with $\alpha \geq 1$. Note that, when V is convex and

$$\mathrm{Hess}(V) \geq \mathrm{cst}\, \mathrm{Id}$$

on the set where $|V| < +\infty$, the measure $M(x)dx$ satisfies the log-Sobolev inequality, which in turn implies (1) (see [Led01]).

In the sequel, by $L^2(\mathbb{R}^n, M)$, we mean the space of measurable functions on $\mathbb{R}^n$ which are square integrable with respect to the measure $M(x)\, dx$, by $L^2_0(\mathbb{R}^n, M)$ the subspace of functions of $L^2(\mathbb{R}^n, M)$ such that $\int_{\mathbb{R}^n} f(x)\, M(x)\, dx = 0$, and by $H^1(\mathbb{R}^n, M)$, the Sobolev space of functions in $L^2(\mathbb{R}^n, M)$, the weak derivative of which belongs to $L^2(\mathbb{R}^n, M)$.

As it shall be proved to be useful later on, remark that, under a slightly stronger assumption than (2), the Poincaré inequality (1) admits the following self-improvement:

Proposition 2.1. *Assume that M there exists $\varepsilon > 0$ such that*

$$\frac{(1-\varepsilon)\, |\nabla V|^2}{2} - \Delta V \xrightarrow[x \to \infty]{} +\infty, \qquad M = e^{-V}. \tag{3}$$

Then there exists $\lambda'(M) > 0$ such that, for all function $f \in L^2_0(\mathbb{R}^n, M) \cap H^1(\mathbb{R}^n, M)$:

$$\iint_{\mathbb{R}^n} |\nabla f(x)|^2\, M(x)\, dx \geq \lambda'(M) \int_{\mathbb{R}^n} |f(x)|^2 \left(1 + |\nabla \ln M(x)|^2\right) M(x)\, dx. \tag{4}$$

We want to generalize the inequality (1) in the strenghtened form of Proposition 2.1, replacing, in the right-hand side, the H^1 semi-norm by a non-local expresssion in the flavour of the Gagliardo semi-norms.

We establish the following theorem:

Theorem 2.1. [MRS11] *Assume that $M = e^{-V}$ is a C^2 positive L^1 function which satisfies* (3)*. Let $\alpha \in (0,2)$. Then there exist $\lambda_\alpha(M) > 0$ and $\delta(M)$ (constructive from our proof and the usual Poincaré constant $\lambda'(M)$) such that, for any function f belonging to a dense subspace of $L^2_0(\mathbb{R}^n, M)$, we have*

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\, M(x)\, e^{-\delta(M)\,|x-y|}\, dx\, dy \geq \tag{5}$$
$$\lambda_\alpha(M) \int_{\mathbb{R}^n} |f(x)|^2 \left(1 + |\nabla \ln M(x)|^\alpha\right) M(x)\, dx.$$

Remark 2.1. *Inequality* (5) *could as usual be extended to any function f with zero average such that both sides of the inequality make sense. In particular it is satisfied for any function f with zero average belonging to the domain of the operator $L = -\Delta - \nabla V \cdot \nabla$ that we shall introduce later on.*

Observe that the left-hand side of (5) involves a fractional moment of order α related to the homogeneity of the semi-norm appearing in the right-hand side. One could expect in the left-hand side of (5) the Gagliardo semi-norm for the fractional Sobolev space $H^{\alpha/2}(\mathbb{R}^n, M)$, namely

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}} M(x)\, M(y)\, dx\, dy.$$

Notice that, instead of this semi-norm, we obtain a "non-symmetric" expression. However, our norm is more natural: one should think of the measure over y as the Lévy measure, and the measure over x as the ambient measure. We emphasize on the fact that our measure is rather general and in particular, as a corollary of Theorem 2.1, we obtain an automatic improvement of the Poincaré inequality (1) by

$$\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\, M(x)\, dx\, dy \geq \lambda_\alpha(M) \int_{\mathbb{R}^n} |f(x)|^2\, M(x)\, dx.$$

The question of obtaining Poincaré-type inequalities (or more generally entropy inequalities) for Lévy operators was studied in the probability community in the last decades.

For instance it was proved by Wu [Wu00] and Chafaï [Cha04] that

$$\mathrm{Ent}_\mu^\Phi(f) \le \int \Phi''(f)\,\nabla f\cdot\sigma\cdot\nabla f\,d\mu + \int\int D_\Phi(f(x),f(x+z))\,d\nu_\mu(z)\,d\mu(x)$$

(see also the use of this inequality in [GI08]) with

$$\mathrm{Ent}_\mu^\Phi(f) = \int \Phi(f)\,d\mu - \Phi\left(\int f\,d\mu\right)$$

and D_Φ is the so-called Bregman distance associated to Φ:

$$D_\Phi(a,b) = \Phi(a) - \Phi(b) - \Phi'(b)\,(a-b),$$

where Φ is some well-suited functional with convexity properties, σ the matrix of diffusion of the process, μ a rather general measure, and ν_μ the (singular) Lévy measure associated to μ. Choosing $\Phi(x) = x^2$ and $\sigma = 0$ yields a Poincaré inequality for this choice of measure (μ,ν_μ). The improvement of our approach is that we do not impose any link between our measure M on x and the singular measure $|z|^{-n-\alpha}$ on $z = x-y$. This is to our knowledge the first result that gets rid of this strong constraint.

Remark 2.2. *Note that the exponentially decaying factor $e^{-\delta(M)\,|x-y|}$ in (5) also* improves *the inequality as compared to what is expected from Poincaré inequality for Lévy measures.*

Our proof heavily relies on fractional powers of a (suitable generalization of the) Ornstein-Uhlenbeck operator, which is defined by

$$Lf = -M^{-1}\,\mathrm{div}(M\,\nabla f) = -\Delta f - \nabla \ln M\cdot\nabla f,$$

for all $f \in \mathcal{D}(L) := \{g \in H^1(\mathbb{R}^n, M);\ \mathrm{div}(M\nabla g) \in L^2(\mathbb{R}^n)\}$. One therefore has, for all $f \in \mathcal{D}(L)$ and $g \in H^1(\mathbb{R}^n, M)$,

$$\int_{\mathbb{R}^n} Lf(x)g(x)M(x)\,dx = \int_{\mathbb{R}^n} \nabla f(x)\cdot\nabla g(x)\,M(x)\,dx.$$

It is obvious that L is symmetric and non-negative on $L^2(\mathbb{R}^n, M)$, which allows to define the usual power L^β for any $\beta\in(0,1)$ by means of spectral theory. Note that $L^{\alpha/2}$ is *not* the symmetric operator associated to the Dirichlet form $\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\alpha}}\,M(x)\,dx\,dy$.

We now describe the strategy of our proofs. The proof of Theorem 2.1 goes in three steps. We first establish L^2 off-diagonal estimates of Gaffney type on the resolvent of L

on $L^2(\mathbb{R}^n, M)$. These estimates are needed in our context since we do not have Gaussian pointwise estimates on the kernel of the operator L.

Then, we bound the quantity

$$\int_{\mathbb{R}^n} |f(x)|^2 \left(1 + |\nabla \ln M(x)|^\alpha\right) M(x)\, dx$$

in terms of $\left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)}$. This will be obtained by an abstract argument of functional calculus based on rewriting in a suitable way the conclusion of Proposition 2.2. Finally, using the L^2 off-diagonal estimates for the kernel of L, we establish that

$$\left\|L^{\alpha/4} f\right\|^2_{L^2(\mathbb{R}^n,M)} \leq C \iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+\alpha}} M(x)\, dx\, dy,$$

which would conclude the proof.

2.2. **The case of Lie groups.** Let G be a unimodular connected Lie group endowed with a measure $M(x)\, dx$ where $M \in L^1(G)$ and dx stands for the Haar measure on G. By "unimodular", we mean that the Haar measure is left and right-invariant. We always assume that M is bounded and $M = e^{-v}$ where v is a C^2 function on G. If we denote by $\mathcal{G}$ the Lie algebra of G, we consider a family

$$\mathbb{X} = \{X_1, ..., X_k\}$$

of left-invariant vector fields on G satisfying the Hörmander condition, i.e. $\mathcal{G}$ is the Lie algebra generated by the $X_i's$. A standard metric on G , called the Carnot-Caratheodory metric, is naturally associated with $\mathbb{X}$ and is defined as follows: let $\ell : [0,1] \to G$ be an absolutely continuous path. We say that ℓ is admissible if there exist measurable functions $a_1, ..., a_k : [0,1] \to \mathbb{C}$ such that, for almost every $t \in [0,1]$, one has

$$\ell'(t) = \sum_{i=1}^{k} a_i(t) X_i(\ell(t)).$$

If ℓ is admissible, its length is defined by

$$|\ell| = \int_0^1 \left(\sum_{i=1}^{k} |a_i(t)|^2\, dt \right)^{\frac{1}{2}}.$$

For all $x, y \in G$, define $d(x,y)$ as the infimum of the lengths of all admissible paths joining x to y (such a curve exists by the Hörmander condition). This distance is left-invariant. For short, we denote by $|x|$ the distance between e, the neutral element of the group and x, so that the distance from x to y is equal to $|y^{-1}x|$.

For all $r > 0$, denote by $B(x,r)$ the open ball in G with respect to the Carnot-Caratheodory distance and by $V(r)$ the Haar measure of any ball. There exists $d \in \mathbb{N}^*$ (called the local dimension of $(G, \mathbb{X})$) and $0 < c < C$ such that, for all $r \in (0,1)$,

$$cr^d \leq V(r) \leq Cr^d,$$

see [NSW85]. When $r > 1$, two situations may occur (see [Gui73]):

- Either there exist $c, C, D > 0$ such that, for all $r > 1$,

  $$cr^D \leq V(r) \leq Cr^D$$

  where D is called the dimension at infinity of the group (note that, contrary to d, D does not depend on $\mathbb{X}$). The group is said to have polynomial volume growth.
- Or there exist $c_1, c_2, C_1, C_2 > 0$ such that, for all $r > 1$,

  $$c_1 e^{c_2 r} \leq V(r) \leq C_1 e^{C_2 r}$$

  and the group is said to have exponential volume growth.

When G has polynomial volume growth, it is plain to see that there exists $C > 0$ such that, for all $r > 0$,

$$V(2r) \leq CV(r), \tag{6}$$

which implies that there exist $C > 0$ and $\kappa > 0$ such that, for all $r > 0$ and all $\theta > 1$,

$$V(\theta r) \leq C\theta^\kappa V(r). \tag{7}$$

Denote by $H^1(G, d\mu_M)$ the Sobolev space of functions $f \in L^2(G, d\mu_M)$ such that $X_i f \in L^2(G, d\mu_M)$ for all $1 \leq i \leq k$. We are interested in L^2 Poincaré inequalities for the measure

$d\mu_M$. In order to state sufficient conditions for such an inequality to hold, we introduce the operator

$$L_M f = -M^{-1} \sum_{i=1}^{k} X_i \Big\{ M X_i f \Big\}$$

for all f such that

$$f \in \mathcal{D}(L_M) := \left\{ g \in H^1(G, d\mu_M);\ \frac{1}{\sqrt{M}} X_i \Big\{ M X_i g \Big\} \in L^2(G, dx),\ \forall 1 \leq i \leq k \right\}.$$

One therefore has, for all $f \in \mathcal{D}(L_M)$ and $g \in H^1(G, d\mu_M)$,

$$\int_G L_M f(x) g(x) d\mu_M(x) = \sum_{i=1}^{k} \int_G X_i f(x) \cdot X_i g(x) d\mu_M(x).$$

In particular, the operator L_M is symmetric on $L^2(G, d\mu_M)$.
Say that a C^2 function $W : G \to \mathbb{R}$ is a Lyapunov function if $W(x) \geq 1$ for all $x \in G$ and there exist constants $\theta > 0$, $b \geq 0$ and $R > 0$ such that, for all $x \in G$,

$$-L_M W(x) \leq -\theta W(x) + b \mathbf{1}_{B(e,R)}(x), \tag{8}$$

where, for all $A \subset G$, $\mathbf{1}_A$ denotes the characteristic function of A. We first claim:

Theorem 2.2. *Assume that G is unimodular and that there exists a Lyapunov function W on G. Then, $d\mu_M$ satisfies the following L^2 Poincaré inequality: there exists $C > 0$ such that, for every function $f \in H^1(G, d\mu_M)$ with $\int_G f(x) d\mu_M(x) = 0$,*

$$\int_G |f(x)|^2 d\mu_M(x) \leq C \sum_{i=1}^{k} \int_G |X_i f(x)|^2 d\mu_M(x). \tag{9}$$

Let us give, as a corollary, a sufficient condition on v for (9) to hold:

Corollary 2.1. *Assume that G is unimodular and there exist constants $a \in (0,1)$, $c > 0$ and $R > 0$ such that, for all $x \in G$ with $|x| > R$,*

$$a \sum_{i=1}^{k} |X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c. \tag{10}$$

Then (9) holds.

Notice that, if (10) holds with $a \in \left(0, \frac{1}{2}\right)$, then the Poincaré inequality (9) admits the following improvement:

Proposition 2.2. *Assume that G is unimodular and that there exist constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that, for all $x \in G$,*

$$\frac{1-\varepsilon}{2}\sum_{i=1}^{k}|X_i v(x)|^2 - \sum_{i=1}^{k} X_i^2 v(x) \geq c \text{ whenever } |x| > R. \tag{11}$$

Then there exists $C > 0$ such that, for every function $f \in H^1(G, d\mu_M)$ such that $\int_G f(x)d\mu_M(x) = 0$:

$$\int_G |f(x)|^2 \left(1 + \sum_{i=1}^{k}|X_i v(x)|^2\right) d\mu_M(x) \leq C \sum_{i=1}^{k}\int_G |X_i f(x)|^2\, d\mu_M(x) \tag{12}$$

Finally, we obtain Poincaré inequalities for $d\mu_M$ involving a non local term.

Theorem 2.3. [RS11] *Let G be a unimodular Lie group with polynomial growth. Let $d\mu_M = M dx$ be a measure absolutely continuous with respect to the Haar measure on G where $M = e^{-v} \in L^1(G)$ is assumed to be bounded and $v \in C^2(G)$.*

1. *Assume that there exist constants $a \in (0,1)$, $c > 0$ and $R > 0$ such that, for all $x \in G$ with $|x| > R$, (10) holds. Then there exists $\lambda_\alpha(M) > 0$ such that, for any function $f \in \mathcal{D}(G)$ satisfying $\int_G f(x)\, d\mu_M(x) = 0$,*

$$\int_G |f(x)|^2\, d\mu_M(x) \leq \lambda_\alpha(M) \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|y^{-1}x|)\,|y^{-1}x|^\alpha}\, dx\, d\mu_M(y). \tag{13}$$

2. *Assume that there exist constants $c > 0$, $R > 0$ and $\varepsilon \in (0,1)$ such that (11) holds. Let $\alpha \in (0,2)$. Then there exists $\lambda_\alpha(M) > 0$ such that, for any function $f \in \mathcal{D}(G)$ satisfying $\int_G f(x)\, d\mu_M(x) = 0$,*

$$\begin{aligned} &\int_G |f(x)|^2 \left(1 + \sum_{i=1}^{k}|X_i v(x)|^2\right)^{\alpha/2} d\mu_M(x) \\ &\leq \lambda_\alpha(M) \iint_{G\times G} \frac{|f(x)-f(y)|^2}{V(|y^{-1}x|)\,|y^{-1}x|^\alpha}\, dx\, d\mu_M(y). \end{aligned} \tag{14}$$

2.3. **The case of Riemannian manifolds.** Let $\mathcal{M}$ be a Riemannian manifold, denote by n its dimension, by $d\mu$ its Riemannian measure and by Δ the Laplace-Beltrami operator. For all $x \in \mathcal{M}$ and all $r > 0$, let $B(x,r)$ be the open geodesic ball centered at x with radius r, and $V(x,r)$ its measure.

In order to apply our method, we will need to be able to control by below the volume of any geodesic ball $B(x,r)$ by a quantity of the type r^p. The goal of the next paragraph is to give sufficient assumptions on M such that this control occurs.

The first one is a Faber-Krahn inequality on $\mathcal{M}$. For any bounded open subset $\Omega \subset \mathcal{M}$, denote by $\lambda_1^D(\Omega)$ the principal eigenvalue of $-\Delta$ on Ω under the Dirichlet boundary condition. If $p \geq n$, consider the following Faber-Krahn inequality: there exists $C > 0$ such that

$$\lambda_1^D(\Omega) \geq C\mu(\Omega)^{\frac{2}{p}} \text{ for all bounded subset } \Omega \subset M. \tag{15}$$

Let $\Lambda_p > 0$ be the greatest constant for which (15) is satisfied. In other words,

$$\Lambda_p = \inf \frac{\lambda_1^D(\Omega)}{\mu(\Omega)^{\frac{2}{p}}},$$

where the infimum is taken over all bounded subsets $\Omega \subset \mathcal{M}$. The Faber-Krahn inequality (15) is satisfied in particular when an isoperimetric inequality holds on $\mathcal{M}$: namely there exists $C > 0$ and $p \geq n$ such that, for all bounded smooth subset $\Omega \subset \mathcal{M}$,

$$\sigma(\partial\Omega) \geq C\mu(\Omega)^{1-\frac{1}{p}}, \tag{16}$$

where $\sigma(\partial\Omega)$ denotes the surface measure of $\partial\Omega$. If $\mathcal{M}$ has nonnegative Ricci curvature, then (16) with $p = n$ and (15) with $p = n$ are equivalent. More generally, if $\mathcal{M}$ has Ricci curvature bounded from below by a constant, (15) with $p > 2n$ implies (16) with $\frac{p}{2}$ ([Car96], Proposition 3.1, see also [Cou92] when the injectivity radius of $\mathcal{M}$ is furthermore assumed to be bounded). Note that there exists a Riemannian manifold satisfying (15) for some $p \geq n$ but for which (16) does not hold for any $p \geq n$ ([Car96], Proposition 3.4).

It is a well-known fact that (15) implies a lower bound for the volume of geodesic balls in $\mathcal{M}$. Namely ([Car96], Proposition 2.4), if (15) holds, then, for all $x \in \mathcal{M}$ and all $r > 0$,

$$V(x,r) \geq \left(\frac{\Lambda_p}{2^{p+2}}\right)^{\frac{p}{2}} r^p. \tag{17}$$

We will also need another assumption on the volume growth of the balls in $\mathcal{M}$, already encountered in the present work in the case of Lie groups . Say that $\mathcal{M}$ has the doubling property if and only if there exists $C > 0$ such that, for all $x \in \mathcal{M}$ and all $r > 0$,

$$(D) \qquad V(x, 2r) \leq CV(x, r).$$

There is a wide class of manifolds on which (D) holds. First, as already said in the introduction (see (6)), it is true on Lie groups with polynomial volume growth (in particular on nilpotent Lie groups). Next, (D) is true if $\mathcal{M}$ has nonnegative Ricci curvature thanks to the Bishop comparison theorem (see [BC64]). Recall also that (D) remains valid if $\mathcal{M}$ is quasi-isometric to a manifold with nonnegative Ricci curvature, or is a cocompact covering manifold whose deck transformation group has polynomial growth, [CSC95]. Contrary to the doubling property, the nonnegativity of the Ricci curvature is not stable under quasi-isometry.

The last assumption we need on $\mathcal{M}$ is a local L^2 Poincaré inequality on balls *for the Riemannian measure*. Namely, if $R > 0$, say that $\mathcal{M}$ satisfies (P_R) if and only if there exists $C_R > 0$ such that, for all $x \in \mathcal{M}$, all $r \in (0, R)$ and all function $f \in C^\infty(B(x, r))$,

$$(P_R) \qquad \int_{B(x,r)} \left|f(x) - f_{B(x,r)}\right|^2 d\mu(x) \leq C_R r^2 \int_{B(x,r)} |\nabla f(x)|^2 \, d\mu(x).$$

Note that on a unimodular Lie group G equipped with vector fields as in the introduction, such a Poincaré inequality always holds. Recall that (P_R) always holds for all $R > 0$ for instance when M has nonnegative Ricci curvature ([Bus82]).

Under these assumptions, the proof developed above in the context of groups, can be adapted *verbatim* to give the following result.

Theorem 2.4. [RS11] *Let $\mathcal{M}$ be a complete non compact Riemannian manifold. Assume that* (15) *holds, that $\mathcal{M}$ has the doubling property and that* (P_R) *holds for some $R > 0$. Let v be a C^2 function on $\mathcal{M}$ and $M = e^{-v}$.*

1. *Assume that there exists $x_0 \in \mathcal{M}$ and constants $a \in (0, 1)$ and $c > 0$ such that, for all $x \in G$ with $d(x, x_0) > R$,*

$$(18) \qquad a \left|\nabla v(x)\right|^2 - \Delta v(x) \geq c.$$

Then, there exists $C>0$ such that, for every function $f\in H^1(\mathcal{M},Md\mu)$ such that $\int_{\mathcal{M}} f(x)M(x)d\mu=0$, for all $\alpha\in(0,1)$,

$$\int_{\mathcal{M}} f^2(x)M(x)d\mu(x)\le C\iint_{\mathcal{M}\times\mathcal{M}}\frac{|f(y)-f(x)|^2}{d(x,y)^{p+\alpha}}M(x)d\mu(x)d\mu(y). \tag{19}$$

2. *Assume there exist $x_0\in\mathcal{M}$ and constants $c>0$ and $\varepsilon\in(0,1)$ such that, for all $x\in\mathcal{M}$,*

$$\frac{1-\varepsilon}{2}|\nabla v(x)|^2-\Delta v(x)\ge c \text{ whenever } d(x,x_0)>R. \tag{20}$$

Then there exists $C>0$ such that, for every function $f\in H^1(\mathcal{M},Md\mu)$ such that $\int_{\mathcal{M}} f(x)M(x)d\mu=0$, for all $\alpha\in(0,1)$,

$$\int_{\mathcal{M}} f^2(x)(1+|\nabla v|^2)^{\alpha/2}M(x)d\mu(x)\le C\iint_{\mathcal{M}\times\mathcal{M}}\frac{|f(y)-f(x)|^2}{d(x,y)^{p+\alpha}}M(x)d\mu(x)d\mu(y). \tag{21}$$

3. An excursion on Q-curvature and fractional Poincaré inequalities

The Q-curvature arises naturally as a conformal invariant associated to the Paneitz operator. When $n=4$, the Paneitz operator is defined as:

$$P_g=\Delta^2+\delta(\frac{2}{3}Rg-2Ric)d,$$

where δ is the divergence, d is the differential, R is the scalar curvature of g, and Ric is the Ricci curvature tensor. The Branson Q-curvature is defined as

$$Q_g=\frac{1}{12}\left\{-\Delta R+\frac{1}{4}R^2-3|E|^2,\right\}$$

where E is the traceless part of Ric, and $|\cdot|$ is taken with respect to the metric g. Under the conformal change $g_u=e^{2u}g_0$, the Paneitz operator transforms by $P_{g_u}=e^{-4u}P_{g_0}$, and Q_{g_u} satisfies the fourth order equation

$$P_{g_0}u+2Q_{g_0}=2Q_{g_u}e^{4u}. \tag{22}$$

This is analogous to the transformation law satisfied by the Laplacian operator $-\Delta_g$ and the Gaussian curvature K_g on surfaces,

$$-\Delta_{g_0}u+K_{g_0}=K_{g_u}e^{2u}.$$

The Q-curvature and the corresponding Paneitz operator P_g are defined for all even dimensions. On a two dimensional manifold, P_g is given by the negative Laplacian operator $-\Delta_g$, and $Q_g = \frac{1}{2}K_g$ the Gaussian curvature of the metric. For higher even dimensions $n \geq 6$, the Q-curvature is defined via the analytic continuation in the dimension and the formula is not explicit in general. However when the background metric is flat, it satisfies, under the conformal change of metric $g_u = e^{2u}|dx|^2$, the n-th order differential equation

$$(-\Delta)^{\frac{n}{2}} u = 2Q_{g_u} e^{nu}, \tag{23}$$

where Δ is the Laplacian operator of $|dx|^2$.

In [Wan15], the second author has studied the isoperimetric inequality by proving the volume form is a strong A_∞ weight on a conformally flat manifold with finite total Q-curvature and nonegative scalar curvature at infinity. The main purpose of [SW17] is to investigate the relation between Q-curvature and Poincare inequality.

Theorem 3.1. [SW17]*Let $(M^n, g) = (\mathbb{R}^n, g = e^{2u}|dx|^2)$ be a complete noncompact even dimensional manifold. Let Q^+ and Q^- denote the positive and negative part of Q_g respectively; and dv_g denote the volume form of M. If*

$$\beta^+ := \int_{M^n} Q^+ dv_g < c_n \tag{24}$$

where $c_n = 2^{n-2}(\frac{n-2}{2})!\pi^{\frac{n}{2}}$,

$$\beta^- := \int_{M^n} Q^- dv_g < \infty, \tag{25}$$

and the scalar curvature at infinity is nonnegative, i.e. $\liminf_{|x|\to\infty} R_g(x) \geq 0$, then (M^n, g) satisfies the fractional Poincaré inequality with constant depending only on n, β^+ and β^-. Namely, for $\alpha \in (0,2)$, there exists $C > 0$ depending only on n, β^+ and β^-, such that for any function f in $C^2(M^n)$ and any Euclidean ball B, $\omega(x) = e^{nu(x)}$

$$\int_B |f(x) - f_{B,\omega}|^2 \omega(x) dx \leq C \int_{2B}\int_{2B} \frac{|f(x) - f(y)|^2}{d_g(x,y)^{n+\alpha}} \omega(x)\omega(y) dx dy. \tag{26}$$

If Ricci curvature is bounded from below, then as an application of the segment inequality by Cheeger-Colding, the Poincaré inequality is known. However, to the best of our knowledge, one cannot derive the Poincaré inequality if only assuming the scalar curvature is nonnegative. There are two intellectual merits of our work. It presents for the first time the relationship between conformal invariants Q-curvature and the Poincaré inequality. It demonstrates that only the integral of the positive and negative part of the Q-curvature would affect the existence of Poincaré inequality, not the distribution of the Q-curvature over the whole manifold.

To prove Theorem 3.1, we first derive the 2-Poincaré inequality by the fact that the volume form e^{nu} is a strong A_∞ weight. This step gives in particular an important geometric meaning of the Q-curvature. As the second step, we rewrite properly this Poincaré inequality and using spectral theory to estimate powers of a suitable weighted Laplacian. Finally, we derive the desired inequality in Theorem 3.1 by a covering argument and some estimates. In this step, we have used the volume growth estimate of geodesic balls which was proved by properties of conformally flat manifolds with finite total Q-curvature. This 2-Poincaré inequality and covering argument approach has been successfully used in [MRS11, RS11] to derive some types of fractional Poincaré inequalities in some Euclidean or geometric contexts. Notice that in these latter works, the fractional Poincaré inequality is not symmetric with respect to the measure in the right hand side. In our theorem, this is the case due to a suitable covering (the covering is with respect to the conformal metric $e^{2u}|dx|^2$) and also the fact that we are considering local estimates.

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Johns Hopkins University, Department of Mathematics, Krieger Hall, Baltimore, MD, USA

E-mail address: sire@math.jhu.edu

Johns Hopkins University, Department of Mathematics, Krieger Hall, Baltimore, MD, USA

E-mail address: ywang@math.jhu.edu