LECTURES IN MATHEMATICS
Department of Mathematics
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LECTURES ON COBORDISM THEORY

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Preface

These are the notes from 6 lectures I gave at Kyoto University in the spring of 1967. They deal with the algebraic problems which arise in the determination of various cobordism theories, especially Spin, Pin, Spin^c, and PL(both oriented and unoriented). The ideas and results are taken from my published and unpublished joint work with D. W. Anderson and E. H. Brown, W. Browder and A. Liulevicius, D. Sullivan, and H. Toda.

F. P. Peterson

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§ 1. Introduction.

First we recall Thom's theory of cobordism. Let O be the orthogonal group and $G \longrightarrow O$ a homomorphism ($G(k) \longrightarrow O(k)$ are suitable homomorphisms for each k): for example we consider the cases $G = O, SO, U, SU, Spin$. There is a map g of the classifying space $BG(k)$ into $BO(k)$ such that for the universal vector bundle γ_k over $BO(k)$, $g^*\gamma_k$ is a universal bundle over $BG(k)$. We denote:

$$\begin{aligned} MG(k) &= \text{Thom space of the bundle } g^*\gamma_k \\ &= \text{one point compactification of the bundle space } E \\ &= E_{\leq 1} / E_{=1}\,. \end{aligned}$$

Always we assume that the coefficient group is Z_2 and is omitted. As is well known we have Thom's isomorphism

$$\phi \,:\, H^*(BG(k)) \cong H^{*+k}(MG(K)).$$

Whitney sum with a trivial line bundle defines a natural map $SMG(k) \longrightarrow MG(k+1)$, hence $\{MG(k)\}$ forms a spectrum $\underline{MG}$, $(\underline{MG})_k = MG(k)$. Then the Thom isomorphism becomes

$$H^*(BG) \overset{\phi}{\cong} H^*(\underline{MG}) \equiv \lim_{k\to\infty} H^{*+k}(MG(k)) \qquad \text{(spectrum cohomology).}$$

Now Thom's first theorem states

<u>Theorem</u>(Thom) $\Omega_n^G \cong \lim_k \pi_{n+k}(MG(k)) \equiv \pi_n(\underline{MG})$.

From now we shall use <u>no geometry</u>. To study homotopy theory of $\underline{MG}$ for various G, the main tool is to study the structure of $H^*(\underline{MG})$

If G has Whitney sums, that is, there are mappings

$$BG(k) \times BG(\ell) \longrightarrow BG(k+\ell)$$

with appropriate properties, then this defines mappings

$$MG(k) \wedge MG(\ell) \longrightarrow MG(k+\ell)$$

and thus a map $\underline{MG} \wedge \underline{MG} \longrightarrow \underline{MG}$ of spectrum. Therefore $H^*(\underline{MG})$ is a coalgebra. Here $\mathcal{A}$ operates on $H^*(\underline{MG}) \otimes H^*(\underline{MG})$ via the Cartan formula.

Case 1. $G = O$

We have the following

Thom's theorem

$H^*(\underline{MO})$ = free $\mathcal{A}$ -module

Therefore $\underline{MO}$ is equivalent to the wedges of $K(Z_2,k)$, the Eilenberg-MacLane spectrum. (Thom gave a long calculational proof)

Case 2. $G = SO$

For this case we have the following

Wall's theorem

$H^*(\underline{MSO})$ = direct sum of $\mathcal{A}/\mathcal{A}(Sq^1) \oplus$ free $\mathcal{A}$ -module and further he proved

$$\underline{MSO} \underset{2}{\sim} \text{wedges of } \underline{K}(Z,k) \text{ and } \underline{K}(Z_2,k) .$$

Before we state the case 3 we give a simpler proof of these theorems.

Proof of Case 1.

Theorem 1. Let M be a connected coalgebra with unit over $\mathcal{A}$, a Hopf algebra. Define a homomorphism $\phi : \mathcal{A} \longrightarrow M$ by $\phi(a) = a(1)$. If $\operatorname{Ker} \phi = 0$, then M is a free $\mathcal{A}$ -module. (This is a theorem due to Milnor Moore)

Proof. We denote by $\bar{\mathcal{A}}$ the positive dimensional elements of $\mathcal{A}$. We set $\bar{M} = M/\bar{\mathcal{A}}\cdot M$, then it is a graded vector space. Let $\pi: M \longrightarrow \bar{M}$ be a projection. Let $\{\bar{m}_i\}$ be a Z_2-basis for $\bar{M}$ such that dim. $\bar{m}_i \leq$ dim. $\bar{m}_{i+1}$. Choose a homomorphism $g : \bar{M} \longrightarrow M$ such that $\pi g = \mathrm{id}$ and $m_i = g(\bar{m}_i)$. We define $\theta : \mathcal{A} \otimes \bar{M} \longrightarrow M$ by $\theta(a \otimes \bar{m}) = a \cdot g(\bar{m})$. Then this is a map of left $\mathcal{A}$-modules. The elements $\{m_i\}$ form a generating set over $\mathcal{A}$ for M. So it is obvious that it is epimorphic. We want to prove that θ is a monomorphism.

Put

$$\bar{M}_n = \bar{M}/ \text{ vector space spanned by } \bar{m}_i,\ i \leq n.$$

We consider the compositions of the following maps:

$$\mathcal{A} \otimes M \xrightarrow{\theta} M \xrightarrow{\psi} M \otimes M \xrightarrow{1 \otimes \pi} M \otimes \bar{M} \longrightarrow M \otimes \bar{M}_n$$

(The last one is a natural projection)

Let $\sum_{i \leq n} a_i \otimes \bar{m}_i \in \mathcal{A} \otimes \bar{M}$ be in Ker. θ with $a_n \neq 0$. $m_n \neq 0$

The element $\sum_{i \leq n} a_i \otimes \bar{m}_i$ is mapped by θ to $\Sigma a_i m_i = 0$ in M. And then it is mapped to $\Sigma \Sigma a_i' m_i' \otimes a_i'' m_i''$ by ψ. $(\psi(a_i) = \Sigma a_i' \otimes a_i'', \psi(m_i) = \Sigma \bar{m}_i' \otimes \bar{m}_i'')$. Then it is mapped to $\Sigma \widehat{a_i m_i'} \otimes m_i''$ (note that $\deg \bar{m}_i \leq \deg \bar{m}_n$), finally to $a_n(1) \otimes \bar{m}_n$ in $M \otimes \bar{M}_n$. Hence $a_n(1) = 0$ and so $a_n = 0$ as $\mathrm{Ker} . \phi = 0$.

This is a contradiction. q. e. d.

By using the same method (but more complicated) we can prove:

Theorem 2'. Let M be a connected coalgebra over $\mathcal{A}$. Let $\phi : \mathcal{A} \longrightarrow M$. Assume $\mathrm{Ker}\, \phi = \mathcal{A}(Sq^1)$. Then $M \cong$ direct sum of copies of $\mathcal{A}/\mathcal{A}(Sq^1) \oplus$ free.

Once we prove this, this implies Wall's theorem. Theorem 2' is a bad theorem, because it does not generalize to the case Ker $\phi = \mathcal{A}(Sq^1, Sq^2)$ (this corresponds to the case $\underline{M}$Spin).

We need some notations.

If X is an $\mathcal{A}$-module, let $Q_0 = Sq^1 \in \mathcal{A}$, then $Q_0^2 = 0$. So Q_0 acts as differential on X. Then we may consider $H(X : Q_0)$.

<u>Theorem 2.</u> Assume given $\theta' : \mathcal{A}/\mathcal{A}(Sq^1) \otimes X \longrightarrow M$ (X is a graded vector space), a map of left $\mathcal{A}$-modules such that

$$\theta^1 : H(\mathcal{A}/\mathcal{A}(Sq^1) \otimes X : Q_0) \longrightarrow H(M ; Q_0)$$

is an isomorphism. (M is connected coalgebra over $\mathcal{A}$, Ker $\phi = \mathcal{A}(Sq^1)$). Then θ' is a monomorphism and $M/ \operatorname{Im} \theta'$ is a free $\mathcal{A}$-module.

Theorem 2 $\Longrightarrow$ Theorem 2'.

<u>Lemma.</u> If N is an $\mathcal{A}$-module then there exists $\theta' : \mathcal{A}/\mathcal{A}(Sq^1) \otimes X \to N$ which is an isomorphism on $H(\ : Q_0)$.

($H(\mathcal{A}/\mathcal{A}(Sq^1) : Q_0) = Z_2$ generated by Sq^0.

Take a basis for $H(N : Q_0)$

$\mathcal{A}/\mathcal{A}(Sq^1) \longrightarrow$ each basis element .)

We set $T = \mathcal{A}/\mathcal{A}(Sq^1) \otimes X$ and let $\pi : M \longrightarrow \bar{M} = M/\bar{\mathcal{A}} \cdot M$ be the projection.

We find $Z \subset M$ such that $\pi | Z$ is a monomorphism and $\bar{M} = \pi(\theta'(T)) \oplus \pi(Z)$. Let $N = T \oplus (\mathcal{A} \otimes Z)$ and $\theta : N \longrightarrow M$, $\theta | T = \theta'$ and $\theta(Z) = Z$. Extend it to $\mathcal{A} \otimes Z$ by linearity.

We prove that θ is isomorphic. Set $N^{(n)}$ = sub $\mathcal{A}$-module generated by N^i, $i \leq n$. In general we have $\theta^{(n)} = N^{(n)} \longrightarrow M^{(n)}$. We prove that

$\theta^{(n)}$ is an isomorphism by induction on n. As before, $\theta^{(n)}$ is an epimorphism (it is obvious by the choice).

$\theta^{(0)}: \mathcal{Q}/\mathcal{Q}(Sq^1) \longrightarrow M^{(0)}$ is an isomorphism by the assumption that $\operatorname{Ker} \phi = \mathcal{Q}(Sq^1)$.

Assume that $\theta^{(n-1)}$ is an isomorphism. Consider the homomorphism

$$\lambda : N/N^{(n-1)} \longrightarrow M/M^{(n-1)}.$$

Lemma $\lambda \mid X^n \oplus Z^n \oplus Sq^1Z^n$ is a monomorphism.

λ induces an isomorphism on $H(\quad :Q_o)$. Here

$$\begin{aligned} H_q(N/N^{(n-1)}:Q_o) &= 0 \qquad \text{for } q < n, \\ &= X^n \qquad \text{for } q = n. \end{aligned}$$

Therefore $\lambda|X^n$ is a monomorphism. So if $\lambda(X_n + Z_n) = 0$, then $\theta(X_n + Z_n) \in M^{(n-1)}$. Therefore by the choice of Z, we have $Z_n = 0$, and hence $X_n = 0$. Finally if $\lambda(Sq^1Z_n) = 0$, then $\theta(Sq^1Z_n) \in (M^{(n-1)})^{n+1}$ and therefore $H(M^{(n-1)}:Q_o) = 0$ in dimension $n + 1$ and n.

We have $\theta(Sq^1Z\ ) = Sq^1(m) \qquad$ for $m \in (M^{(n-1)})^n$

$m = \theta(y) \qquad$ for $y \in (N^{(n-1)})^n$

So $Sq^1\theta(Z_n + y) = 0$, therefore $\theta(Z_n + y) = m'$, $m' \in M^{(n-1)}$. By choice of Z we obtain $Z_n = 0$ and hence $Sq^1Z_n = 0$. (This is the same argument as before.)

Conclusion of proof

We want to prove that λ on $N^{(n)}/N^{(n-1)}$ is a monomorphism.

Let $\{v_i\}$ be a basis for $X^n \oplus Z^n \oplus Sq^1Z^n$. Then $v \in N^{(n)}/N^{(n-1)}$ is of the form

$$v = \Sigma a_i v_i \quad \text{with} \quad a_i \notin \mathcal{A}(Sq^1)$$

Assume $v \neq 0$, $\lambda(v) = 0$. Consider the compositions of the following homomorphisms $N/N^{(n-1)} \longrightarrow M/M^{(n-1)} \longrightarrow M \otimes M/M^{(n-1)}$

Then v is mapped to 0 in $M/M^{(n-1)}$ and then to $\Sigma\, a_i(1) \otimes \lambda(v_i)$ + (terms in different dimensions) in $M \otimes M/M^{(n-1)}$.

Therefore $\phi(a_i) = a_i(1) = 0$. Hence $a_i \in \mathcal{A}(Sq^1)$ for all i. This is a contradiction.

Let me state Theorem 3 without proof. One can prove the following theorem by a similar but much more complicated method.

Theorem 3. Let M be a connected, coalgebra over $\mathcal{A}$. Assume $\operatorname{Ker}\phi = \mathcal{A}(Sq^1, Sq^2)$. Let X and Y be graded vector spaces. Assume that $\theta' : \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \otimes X \oplus (\mathcal{A}/\mathcal{A}(Sq^3) \otimes Y) \longrightarrow M$ is an isomorphism on $H(\quad : Q_0)$ and $H(\quad : Q_1)$, then θ' is a monomorphism and $M/\operatorname{Im}\theta'$ is free. (Here $Q_1 = Sq^3 + Sq^2 Sq^1$ and $Q_1{}^2 = 0$).

Its application is for $H^*(\underline{M}Spin) = M$.

This is not the most general theorem, but it works in the application.

From Theorem 3, one could calculate $\pi_*(\underline{M}Spin)$ by applying the Adams spectral sequence.

That is, one calculates

$$\operatorname{Ext}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}(Sq^1, Sq^2), Z_2),$$

$$\operatorname{Ext}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}(Sq^3), Z_2),$$

and then show $E_2 = E_\infty$ (for algebraic reasons).

We find a spectrum $\underline{X}$ whose cohomology is $\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$

and another spectrum $\underline{Y}$ whose cohomology is $\mathcal{A}/\mathcal{A}(Sq^3)$:

$$\underline{\mathrm{MSpin}} \longrightarrow \vee\, \underline{X} \,\vee\, \underline{Y} \,\vee\, \underline{K}(Z_2,\quad)$$

Let $BO\langle n\rangle = BO(n,\ldots,\infty) =$ (n-1)-connective fibering of BO. We have the map $p : BO\langle n\rangle \longrightarrow BO$. Then

$$p_*:\pi_*(BO\langle n\rangle) \longrightarrow \pi_*(BO) \text{ is isomorphic if } * \geq n,$$
$$\text{is zero if } * < n.$$

By Bott we have $BO = \Omega^{8\infty}(BO)$.

One can find a Ω-spectrum $\underline{BO}\langle n\rangle$ with $(\underline{BO}\langle n\rangle)_0 = BO\langle n\rangle$. Then we have

<u>Theorem(Stong)</u>

$$H^*(\underline{BO}\langle n\rangle) = \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \quad \text{if } n \equiv 0(8),$$
$$= \mathcal{A}/\mathcal{A}(Sq^3) \quad \text{if } n \equiv 2(8).$$

§ 2. Results about Spin cobordism.

I want to describe the Spin cobordism Ω_*^{Spin} .

$BSpin \longrightarrow BSO$ is the 2-connective fibering. You take $\pi_2(BSO) \cong Z_2$. Kill it, then you get BSpin. Classically, $Spin(k) \longrightarrow SO(k)$ is a 2-fold covering space. Then you have that $MSpin(k)$ forms spectrum $\underline{MSpin}$ and $\pi_*(\underline{MSpin}) = \Omega_*^{\mathrm{Spin}}$.

The cohomology $H^*(BSpin)$ is easy to compute from the fibering $BSpin \longrightarrow BSO$ and we obtain Easy Theorem

$$H^*(BSpin) \cong Z_2[w_i],\ i \neq 2^r + 1 \quad \text{as algebra}$$

$$\cong Z_2[w_4, w_6, w_7, w_8, w_{10}, \ldots] .$$

But w_{2^r+1} is not necessarily zero, only decomposable. For example

$$w_5 = 0$$

$$w_9 = 0$$

$$w_{17} = w_4 \cdot w_{13} + w_7 \cdot w_{10} + w_6 \cdot w_{11}$$

w_{33} has about 200 polynomial terms.

We have that

$$H^*(BSpin) \cong H^*(BSO)/\text{Ideal generated by } w_2,\ Sq^1 w_2,\ Sq^2 Sq^1 w_2, \ldots,\ Sq^{2^{r-1}} Sq^{2^{r-1}} \ldots Sq^1(w_2), \ldots$$

This is an isomorphism as an algebra over $\mathcal{A}$.

(e.g. $Sq^1 w_{16} = w_{17} =$ decomposable)

Before we state the main theorem we need some notations.

Let $J = (j_1, \ldots, j_k)$ be a partition such that $\Sigma j_i = n(J)$,$k \geq 0$

and $j_i > 1$.

Let X be a graded vector space with one generator X_J in dim. $4n(J)$ for each J with $n(J)$ even.

Let Y be a graded vector space with one generator Y_J in dim. $4n(J)-2$ for each J with $n(J)$ odd.

<u>The Main Theorem</u>

$$H^*(\underline{MSpin}) \cong (\mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \otimes X) \oplus (\mathcal{A}/\mathcal{A}(Sq^3) \otimes Y) \oplus (\mathcal{A} \otimes Z)$$

as an $\mathcal{A}$-module

where Z is a graded vector space.

Furthermore there exists elements $\pi^i \in KO(\underline{MSpin})$. (These are images of the KO-Thom isomorphism for Spin-bundles

$$KO^\circ(BSO) \longrightarrow KO^\circ(BSpin) \cong \widetilde{KO}(\underline{MSpin})$$

For reference, see " SU-cobordism, KO-characteristic numbers, and the Kervarire invariant ", Ann. of Math. (1966).

For such an element J we have

$$\pi^J = \pi^{j_1} \cdot \pi^{j_2} \dots \pi^{j_k} \in \widetilde{KO}(\underline{MSpin})$$

We have another theorem.

<u>Theorem</u> Filtration $\pi^J = 4n(J)$ if $n(J)$ even

$= 4n(J) - 2$ if $n(J)$ odd.

Therefore π^J defines a map

$$\pi^J : \underline{MSpin} \longrightarrow \underline{BO}\langle 4n(J)\rangle ,$$

$$\text{or } \underline{BO}\langle 4n(J)-2\rangle ,$$

where $BO\langle n\rangle \longrightarrow BO$ is $(n-1)$-connective fibering. We have a map

$$F : \underline{MSpin} \longrightarrow \bigvee_{\substack{n(J)\\ \text{even}}} \underline{BO}\langle 4n(J)\rangle \vee \bigvee_{\substack{n(J)\\ \text{odd}}} \underline{BO}\langle 4n(J)-2\rangle \vee \bigvee \underline{K}(Z_2, \dots)$$

and the map F induces

$$H^*(\underline{MSpin}) \underset{F^*}{\cong} (\mathcal{A}/\mathcal{A}(Sq^1, Sq^2)\otimes X) \oplus (\mathcal{A}/\mathcal{A}(Sq^3)\otimes Y) \oplus (\mathcal{A}\otimes Z).$$

We will not discuss the KO-theory here. But we will discuss the main theorem.

From this one reads off $\pi_*(\underline{MSpin}) \cong \Omega_*^{Spin}$. Let me give some examples of J. The lowest dimensional J with $n(J)$ even and all integers in J not even is $J = (3, 3)$, $4n(J) = 24$. Milnor, in his study of Ω_*^{Spin}, stopped at 23 because of this element.

We can describe the manifold representing each class except for these of this type, that is, $n(J)$ even and not all integers in J even. There exists a manifold M^{24} with $w_6^4(M^{24}) \neq 0$. We cannot construct M^{24}. It would be interesting problem to find this large class of Spin-manifolds. All other representative manifolds of cobordism classes are constructed by using Dold's manifold etc.

Let me now state the corollaries of the main theorem.

Corollary of the main theorem

1. Let $[M] \in \Omega_*^{Spin}$. Then

$[M] = 0$ if and only if all KO-characteristic numbers and all Stiefel-Whitney numbers vanish. (This is easy from the second theorem.)

2. $\mathrm{Im}\,(\Omega_*^{Spin} \longrightarrow \mathfrak{N}_*)$ = all [M] all of whose Stiefel-Whitney numbers involving w_1 or w_2 vanish.

(I will discuss the proof in details later)

Milnor showed that $\mathrm{Im}(\Omega_*^{Spin} \longrightarrow \mathfrak{N}_*)$ = squares of oriented manifolds in dim. ≤ 23. In general, $\mathrm{Im}(\Omega_*^{Spin} \longrightarrow \mathfrak{N}_*) \supset$ squares of oriented manifolds $\neq$ in dim. 24.

3. $\mathrm{Im}(\Omega_n^{fr} \longrightarrow \Omega_n^{Spin}) \cong Z_2 \qquad n \equiv 1,2\ (8),$

$\qquad\qquad 0 \qquad$ otherwise.

The representative manifold is $[M^8]^k \times S^1$, $[M^8]^k \times S^1 \times S^1$.

(This is not difficult corollary.)

Cf. $\mu_o = \eta$, $\mu_1 = \{8\sigma, 2\iota, \eta\}$, $\mu_k = \{8\sigma, 2\iota, \mu_{k-1}\}$

and $\mu_k \longrightarrow [M^8]^k \times S^1$.

4. (Corollary of 3) The Kervaire-Arf invariant

$$\Phi : \pi_{8k+2}(\$) \longrightarrow Z_2 \text{ is zero if } k \geq 1.$$

Outline of proof:

$$\pi_{8k+2}(\$) \longrightarrow \Omega_{8k+2}^{Spin} \longrightarrow Z_2$$

$$\begin{aligned}\Phi(([M^8]^k \times S^1) \times S^1) &= \Phi(N^{8k+1} \times S^1) \\ &= \Phi(\Sigma^{8k+1} \times S^1) \\ &= \Phi(\Sigma^{8k+2}) = 0.\end{aligned}$$

Now we discuss the algebra needed in the proof of the main theorem.

Let M be a left(right) $\mathcal{A}$-module ($\mathcal{A}$: Steenrod algebra).

Then $M^* = \mathrm{Hom}(M_1, Z_2)$ is a right(left) $\mathcal{Q}$-module by $(m^*)a \cdot m_1 = m^* \cdot a(m_1)$, $m_1 \cdot a(m^*) = (m_1 a) \cdot m^*$. The operators of $\mathcal{Q}$ lower degrees. $\mathcal{Q}$ itself is a left and a right $\mathcal{Q}$-module by multiplication. Therefore $\mathcal{Q}^*$ is a right and left $\mathcal{Q}$-module.

By Milnor's notation, let $\xi_k \in \mathcal{Q}^{*2^k-1}$. Milnor proved that

$$\mathcal{Q}^* = Z_2[\xi_1, \xi_2, \ldots] \qquad \text{as an algebra.}$$

Proposition $\mathcal{Q}^*$ is a left and a right algebra over $\mathcal{Q}$, (Cartan formula holds) and

$$Sq(\xi_k) = \xi_k + \xi_{k-1}^2$$

$$(\xi_k)(Sq) = \xi_k + \xi_{k-1}, \text{ where } Sq = \sum_{i \geq 0} Sq^i .$$

Proof Exercise for the reader.

DK

§ 3. Outline of the proof of the main theorem in § 2.

In order to prove the main theorem we must study $\mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2)$ and also $H(\mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2), Q_0)$, $H(\mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2), Q_1)$. Consider

$$\mathcal{Q} \oplus \mathcal{Q} \xrightarrow{R(Sq^1) \oplus R(Sq^2)} \mathcal{Q} \longrightarrow \mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2) \longrightarrow 0.$$

Dualizing

$$\mathcal{Q}^* \oplus \mathcal{Q}^* \xleftarrow{L(Sq^1) \oplus L(Sq^2)} \mathcal{Q}^* \longleftarrow (\mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2))^* \longleftarrow 0.$$

Applying χ

$$\mathcal{Q}^* \oplus \mathcal{Q}^* \xleftarrow{R(Sq^1) \oplus R(Sq^2)} \mathcal{Q}^* \longleftarrow \chi(\mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2))^* \longleftarrow 0.$$

Let $A = Z_2[\xi_1^4, \xi_2^2, \xi_3 \ldots] \subset \mathcal{Q}^*$.

We have

$$(\xi_k)Sq^1 = 0 \qquad \text{unless } k = 1$$

$$(\xi_1)Sq^1 = \xi_0 = 1$$

$$(\xi_k)Sq^2 = 0 \qquad \text{unless } k = 2$$

$$(\xi_2)Sq^2 = \xi_1 .$$

Also note : $(\xi_1^2)Sq^2 = \xi_0^2 = 1$.

It is easy to prove that

$$A \subset \mathrm{Ker.}(R(Sq^1) + R(Sq^2))$$

$\mathcal{Q}^* =$ free A-module on generators $1, \xi_1, \xi_1^2, \xi_2, \xi_1^3, \xi_1\xi_3, \xi_1^2\xi_2$. $\xi_1^3\xi_2$

2ξ ξ

Therefore the kernel has nothing more than A.

Theorem $\chi(\mathcal{A} / \mathcal{A}(Sq^1, Sq^2))^* = Z_2[\xi_1^4, \xi_2^2, \xi_3, \ldots]$.

Theorem

$$H(\mathcal{A} / \mathcal{A}(Sq^1, Sq^2) : Q_i) \qquad i = 0, 1$$

$$= H(\chi(\mathcal{A} / \mathcal{A}(Sq^1, Sq^2))^* : Q_i)$$

$$= \begin{cases} Z_2[\xi_1^4] & \text{with respect to } Q_0 \\ E(\xi_2^2, \xi_3^2, \xi_4^2, \ldots) & \text{with respect to } Q_1 = Sq^3 + Sq^2 Sq^1. \end{cases}$$

Therefore you can read off

Theorem A basis for $H(\mathcal{A} / \mathcal{A}(Sq^1, Sq^2), Q_0)$ is $\chi(Sq^{4k})$.

Similarly

$$\mathcal{A} \xrightarrow{R(Sq^3)} \mathcal{A} \longrightarrow \mathcal{A} / \mathcal{A}(Sq^3) \longrightarrow 0.$$

$$\mathcal{A}^* \xleftarrow{L(Sq^3)} \mathcal{A}^* \longleftarrow (\mathcal{A} / \mathcal{A}(Sq^3))^* \longleftarrow 0.$$

$$\mathcal{A}^* \xleftarrow{R(Sq^2, Sq^1)} \mathcal{A}^* \longleftarrow \chi(\mathcal{A} / \mathcal{A}(Sq^3))^* \longleftarrow 0.$$

You come up with

Theorem $\chi(\mathcal{A} / \mathcal{A}(Sq^3))^*$ = a free A-module with generators

$1, \xi_1, \xi_1^2, \xi_1^3 + \xi_2, \xi_1 \xi_2$.

Theorem $H(\chi(\mathcal{A}/\mathcal{A}(Sq^3))^* : Q_0) = \xi_1^2 \cdot Z_2[\xi_1^4]$.

$$H(\chi(\mathcal{A}/\mathcal{A}(Sq^3))^* : Q_1) = \xi_1^2 \cdot E(\xi_2^2, \xi_3^2, \dots).$$

In order to apply the techniques of the last time we must study $H(H^*(MSpin) : Q_i)$ $(i = 0,1)$.

Remember the Thom isomorphism that

$$\phi : H^*(BSpin) \longrightarrow H^*(\underline{MSpin})$$

is a map of Q_0 and Q_1 modules, because $Q_0(U) = Q_1(U) = 0$.

Let $B = H^*(BSpin)$ for simplicity.

We recall that

$$B = Z_2[w_i] \qquad i \neq 2^r + 1$$

$$Q_0(w_{2i}) = w_{2i+1} \qquad Q_0(w_{2i+1}) = 0.$$

$$Q_0(w_{16}) = w_{17} = w_4 \cdot w_{13} \quad \dots \qquad (\text{cf. } \phi(w_{16}) = Sq^{16}U).$$

Define $X_i \in B^{2^i}$ by $\phi(X_i) = \chi(Sq^{2^i})(\phi(1))$.

Then $X_i = w_{2i} + \text{decomp.}$ Furthermore $Q_0(X_i) = 0$.

Now we have

$$B = Z_2[X_i, w_j] \qquad j \neq 2^r,\ j \neq 2^r + 1.$$

Furthermore

$$Q_0(w_{2j}) = w_{2j+1} \qquad j \neq 2^r$$

$$Q_0(X_i) = 0$$

We have

$$H(B:Q_0) = Z_2[X_i, (w_j)^2], \quad j \neq 2^r,$$

where $(w_{2j})^2 = p_j$ is a Pontrjagin class. Similarly for Q_1 -case, but $H(B:Q_1)$ is more complicated.

Remember the theorem of last time:

If given θ': $\mathcal{Q}/\mathcal{Q}(Sq^1,Sq^2) \otimes X \oplus \mathcal{Q}/\mathcal{Q}(Sq^3) \otimes Y \longrightarrow H^*(\underline{MSpin})$ such that θ'_* is isomorphic on $H(\quad : Q_i)$, $i = 0,1$, then θ' is monomorphic and cokernel θ' is free $\mathcal{Q}$ -module.

Two difficulties yet arise ; that is,

1. To find θ''
2. To show that θ'_* is isomorphic.

Let X be a graded vector space over X_J.

We would like to send

$$\theta(X_J) = P_J = p_{j_1} \cdot p_{j_2} \cdots\cdots p_{j_k}.$$

$$p_j = (w_{2j})^2, \quad \text{so} \quad Sq^1(p_j) = 0.$$

$$Sq^2(w_{2j})^2 = (w_{2j+1})^2 \neq 0.$$

$$Q_0(p_j) = 0, \; Q_1(p_j) = 0.$$

The results of KO-theory computations show that for $n(J)$ even, there is an element X_J such that $X_J \equiv P_J \mod Q_0Q_1$, that is, $\{ X_J \} = \{ P_J \}$ in $H(\quad : Q_i)$, $i = 0,1$, and $Sq^1(X_J) = 0$, $Sq^2(X_J) = 0$.

If $n(J)$ is odd, there is a class Y_J such that $Sq^2(Y_J) = P$.

(Hence $Sq^3(Y_J) = 0$.)

Define θ' by $\theta'(X_J) = X_J$

$$\theta'(Y_J) = Y_J.$$

To show that θ'_* is isomorphic, we need four more pages of computation.

From the theorem of the last time we obtain the main theorem.

§ 4. The mixed homology.

Let $\mathcal{Q}_1 = \{Sq^0, Sq^1, Sq^2\}$ be the subalgebra of $\mathcal{Q}$. So $Q_0, Q_1 \in \mathcal{Q}_1$, where $Q_1 = Sq^3 + Sq^2Sq^1$, $Q_0 = Sq^1$. If M is an $\mathcal{Q}_1$-module, we can define $H(M;Q_i)$, $i = 0,1$.

We want to define the mixed homology. I also define:

$$(\mathrm{Ker}\ Q_0 \cap \mathrm{Ker}\ Q_1)/(\mathrm{Im}\ Q_0 \cap \mathrm{Im}\ Q_1) \xrightarrow{\eta_i} H(M;Q_i) \qquad i = 0,1$$

<u>Definition</u> M has isomorphic homologies if η_i is isomorphism for $i = 0,1$.

<u>Theorem</u> (Wall)

If $H(M : Q_i) = 0$, then M = free $\mathcal{Q}_1$-module.

A generalization of this is the following

<u>Theorem</u> If M has isomorphic homologies, then M is isomorphic to the direct sums of four types of $\mathcal{Q}_1$-modules, $\mathcal{Q}_1$, $\mathcal{Q}_1/\mathcal{Q}_1(Sq^3)$, $\mathcal{Q}_1/\mathcal{Q}_1(Sq^1, Sq^3)$, Z_2.

The reason I give this theorem is that it is useful in the KO-theory computations which show the existence of X_J and Y_J, $H^*(BSO)$ has isomorphic homologies, so this gives the $\mathcal{Q}_1$-structure of $H^*(BSO)$.

Remember

$$E_1 = \{Sq^0, Q_0, Q_1\} = E(Q_0, Q_1) \subset \mathcal{Q}_1 \subset \mathcal{Q}.$$

The following is easy to prove.

<u>Proposition</u> M, an $\mathcal{Q}_1$-module, has isomorphic homologies

$\Longleftrightarrow$ $M \underset{E_1}{\cong}$ a free E_1-module $\oplus$ a trivial E_1-module.

Let me outiline the proof.

Let $M^{(n)}$ = sub $\mathcal{Q}_1$-module generated by M^i, $i \leq n$.

The proof is done by induction on n.

For $M^{(0)}$, the theorem is true by one page of easy calculation. Consider the sequence $0 \longrightarrow M^{(n-1)} \longrightarrow M \longrightarrow M/M^{(n-1)} \longrightarrow 0$.

First we prove that $M/M^{(n-1)}$ has isomorphic homologies using the alternative definition of isomorphic homologies as E_1-modules (the five lemma does not work, because the degrees of the two differentials are different). Now look at the sequence

$$0 \longrightarrow M^{(n-1)} \longrightarrow M^{(n)} \longrightarrow M^{(n)}/M^{(n-1)} \longrightarrow 0,$$

where $M^{(n)}/M^{(n-1)} = (M/M^{(n-1)})^{(n)}$. Here $M^{(n)}/M^{(n-1)}$ satistfies the conclusion by the same proof as for $M^{(0)}$, so does $M^{(n-1)}$, and one must prove that the extension is trivial. (This takes the $1\frac{1}{2}$ pages of computation).

Let me make one remark : We want the filtration of elements in $KO^0(BSO)$. ($KO^0(BSO)$ is known.) One studies the so-called Atiyah-Hirzebruch spectral sequence from $H^*(BSO : KO^*(pt))$ to $KO^0(BSO)$.

The differentials d_2, d_3, d_4, d_5 are all primary operations in $\mathcal{A}_1$. So knowing $H^*(BSO)$ as an $\mathcal{A}_1$-module and E_∞ allows you to compute the filtrations. (Later I'll say more of $\mathcal{A}_1$-modules.)

Now I want to discuss the problem related to

$$\mathrm{Im}(\Omega_*^{\mathrm{Spin}} \longrightarrow \mathcal{N}_*) = \mathrm{Im}(\pi_*(\underline{\mathrm{MSpin}}) \longrightarrow \pi_*(\underline{\mathrm{MO}})).$$

§ 5. General theory on maps of spectra.

Let $\underline{f} : \underline{X} \longrightarrow \underline{Y}$ be a map of spectra.

Assume always that $\underline{Y} = \vee \underline{K}(Z_2, \ldots\ldots)$.

Question is to describe $\mathrm{Im}(\pi_*(\underline{X}) \xrightarrow{\underline{f}_*} \pi_*(\underline{Y}))$.

Let G_* be a subset of $\pi_*(\underline{Y})$ defined by

$$G_* = \{g : \underline{S} \longrightarrow \underline{Y} \mid g^*(u) = 0 \text{ for all } u \in H^*(\underline{Y}) \text{ with } u \in \mathrm{Ker}\, \underline{f}^*\}.$$

In general, $\mathrm{Im}\, \underline{f}_* \subset G_*$.

When is $\mathrm{Im}\, \underline{f}_* = G_*$?

Definition $\underline{X}$ has a property P

$\Longleftrightarrow$ given $u \in H^*(\underline{X})$ such that $0 \neq u \in H^*(\underline{X})/\bar{\mathcal{A}}\cdot H^*(\underline{X})$ then there exists $g \in \pi_*(\underline{X})$ such that $g^*(u) \neq 0$.

(For example, $\underline{Y}$ has property P.)

Theorem Assume that $\underline{f}^* : H^*(\underline{Y}) \longrightarrow H^*(\underline{X})$ is epimorphic, then $\mathrm{Im}\, \underline{f}_* = G_*$ if and only if $\underline{X}$ has a property P.

Proof ($\Leftarrow$) Let $g : \underline{S} \longrightarrow \underline{Y}$ and $g \in G_* - \mathrm{Im}\, \underline{f}_*$.

That means there exists $u \in H^*(\underline{Y})$ such that $g^*(u) \neq 0$, $(\underline{f}\, g')^*(u) = 0$ for all $g' \in \pi_*(\underline{X})$.

Therefore $g'^*(\underline{f}^*(u)) = 0$ for all g'.

So $\underline{f}^*(u) \in \bar{\mathcal{A}}\cdot H^*(\underline{X})$, whence $\underline{f}^*(u) = a\cdot\underline{f}^*(v)$ dim. $a > 0$

for $u + av \in \mathrm{Ker}\, \underline{f}^*$.

So $g^*(u + av) = 0 = g^*(u)$. This is a contradiction.

ok ($\Longrightarrow$) Let $0 \neq u \in H^*(\underline{X})/\bar{\mathcal{Q}} H^*(\underline{X})$. If $(g')^*(u) = 0$ for all g', $u = \underline{f}^*(v)$, $v \notin \bar{\mathcal{Q}} \cdot H^*(\underline{Y})$, then there exists $g \in \pi_*(\underline{Y})$ such that $g^*(v) \neq 0$ and $g^*(\text{Ker } \underline{f}^*) = 0$.

Therefore $g \in G_* - \text{Im } \underline{f}_*$: contradiction.

Below we give some corollaries of this theorem. Before it, we need a

<u>Proposition</u> If $g : \underline{S} \longrightarrow \underline{MO}$, $g^*(U\cdot(\text{ideal generated by } w_1 \text{ and } w_2)) = 0$, then $g^*(U\cdot(\text{ideal over } \mathcal{Q} \text{ generated by } w_1 \text{ and } w_2)) = 0$.

<u>Proof</u> Let $g : \underline{S} \longrightarrow \underline{MO}$ such that $g^*(U.w_j.w) = 0$ for $j = 1, 2$. We want to prove $g^*(U.a(w_j)\cdot w) = 0$ for all a and w.
This is done by induction on dim.a.

By the Cartan formula we have

$$U\cdot a(w_j)\cdot w = a(U\cdot w_j\cdot w) + \Sigma U\cdot a'(w_j)\cdot w' \text{ , where dim.}a' < \text{dim.}a \text{ .}$$

By induction hypothesis

$$g^*(U\cdot a(w_j)\cdot w) = g^*(a(U\cdot w_j\cdot w) + \Sigma U\ a'(w_j)\cdot w') = 0.$$

Now we get

<u>Theorem</u> $\text{Im}(\Omega_*^{\text{Spin}} \longrightarrow \mathcal{N}_*)$ = all cobordism classes all of whose Stiefel-Whitney numbers involving w_1, or $w_2 = 0$.

ok

<u>Proof</u> The part $\mathrm{Im}(\Omega_*^{\mathrm{Spin}} \longrightarrow \mathcal{N}_*) \subset$ all..... is clear.

Let $g : \underline{S} \longrightarrow \underline{MO}$, then $g(\mathrm{Ker}\ \underline{f}^*) = 0$

then $g \in G_*$.

So we must prove that $\underline{X} = \underline{MSpin}$ has a property P in order to apply the theorem.

<u>Lemma</u> If $E_2 = E_\infty$ in the Adams spectral sequence for $\pi_*(\underline{X})$, then $\underline{X}$ has a property P.

We have $E_2 = E_\infty$ in the case $\underline{X} = \underline{MSpin}$.

Therefore $G_* = \mathrm{Im}\ f^*$.

§ 6. The bordism group.

We also have the bordism " homology " groups.

e.g, $\mathcal{N}_*(K) = \{ (M, f) \mid f : M^n \longrightarrow K \} / \sim$

where $(M_1, f_1) \sim (M_2, f_2)$ if and only if there exists a cobordism W between M_1 and M_2 and a map F such that $F|M_1 = f_1$ and $F|M_2 = f_2$.

Then $\mathcal{N}_*(\text{point}) = \mathcal{N}_*$.

We have another definition due to G. W. Whitehead

$$\mathcal{N}_*(K) = \pi_*(K^+ \wedge \underline{MO}) .$$

We have characteristic numbers for bordism groups. Let $u \in H^{n-k}(K)$ and $w \in H^k(BO)$, then we define

$$<f^*(u) \cdot \nu^*(w), [M^n]> \in Z_2.$$

These are called the characteristic numbers of (M, f). It is easy to prove that $[(M, f)] = 0$ if and only if all characteristic numbers are zero.

<u>Theorem</u> $[\mathrm{Im}(\Omega_*(K) \longrightarrow \mathcal{N}_*(K)) =$ all bordism classes all of whose characteristic numbers (of the map) involving w_1 vanish] holds if and only if $H_*(K : Z)$ has no 4-torsion.

The proof depends on the fact that $K \wedge \underline{MSO}$ has a property P if and only if $H(K : Z)$ has no 4-torsion. (This is easy to prove.)

<u>Theorem</u> There exists a PL-manifold M^9 such that all characteristic numbers if M^9 involving w_1, are zero but $M^9 \not\sim$ orientable PL-manifold.

<u>Theorem</u> [$\mathrm{Im}(\Omega_*^{\mathrm{Spin}}(K) \longrightarrow \mathfrak{N}_*(K)) =$ all bordism classes all of whose characteristic numbers involving w_1 or w_2 vanish] holds if and only if $K \wedge \underline{MSpin}$ has a property P.

Later I will prove that $BSO \wedge \underline{MSpin}$ and $RP^\infty \wedge \underline{MSpin}$ have property P. So this is true for $K = BSO$ and $K = RP^\infty$.

We discuss the methods for computing $\Omega_*^{\mathrm{Spin}}(K)$, $KO_*(K)$ etc.

Recall

$$\Omega_*^{\mathrm{Spin}}(K) = H_*(K : \underline{MSpin}) = \pi_*(K^+ \wedge \underline{MSpin}).$$

One method for computing $H_*(K : \underline{M})$ is the usual spectral sequence :

$$E^2_{**} = H_*(K : \pi_*(\underline{M})) \Longrightarrow E^\infty.$$

Another method is to compute $\pi_*(K \wedge \underline{MSpin}) = \Omega_*^{\mathrm{Spin}}(K)$ using the Adams spectral sequence. That is, one must compute $H^*(K \wedge \underline{MSpin})$ as a module over $\mathcal{A}$, and then apply the Adams spectral sequence.

Here we have

$$\begin{aligned} H^*(K \wedge \underline{MSpin}) &\cong H^*(K) \otimes H^*(\underline{MSpin}) \\ &\cong H^*(K) \otimes (\Sigma\, \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \oplus \Sigma\, \mathcal{A}/\mathcal{A}(Sq^3) \oplus \Sigma\, \mathcal{A}). \end{aligned}$$

So it is enough to study the $\mathcal{A}$-module structure of $M \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$, $M \otimes \mathcal{A}/\mathcal{A}(Sq^3)$ and $M \otimes \mathcal{A}$ for some given M.

$M \otimes \mathcal{A} / \mathcal{A}(Sq^1, Sq^2)$ is the tensor product in the category of $\mathcal{A}$-modules, so by the Cartan formula we have

$$a(m \otimes b) = \Sigma a'm \otimes a''b.$$

<u>Theorem</u> $M \otimes \mathcal{A}$ is a free $\mathcal{A}$-module.

<u>Proof</u> We need some notations :

$\hat{M}$ = underlying Z_2-vector space of M as trivial $\mathcal{A}$-module : $Sq^0 = id$, $Sq^i = 0$ for $i > 0$.

We can form $\hat{M} \otimes \mathcal{A}$ by defining

$$a(m \otimes b) = m \otimes ab \quad \text{for} \quad \dim a > 0.$$

Let us define

$$\ell : \hat{M} \otimes \mathcal{A} \longrightarrow M \otimes \mathcal{A}$$

by $\ell(m \otimes 1) = m \otimes 1$ and extend as an $\mathcal{A}$-map, that is, $\ell(m \otimes a) = \ell a(m \otimes 1) = a\ell(m \otimes 1) = a(m \otimes 1) = \Sigma a'(m) \otimes a''$. This is an $\mathcal{A}$-map.

We prove that ℓ is an isomorphism.

Note that $m \otimes 1 \in \mathrm{Im}.\ell$. Assume $m \otimes a \notin \mathrm{Im}.\ell$. with dim.a minimal. Then $a(m \otimes 1) = \Sigma a'm \otimes a'' = \Sigma a'(m) \otimes a'' + m \otimes a$

$$\dim a'' < \dim a'$$

where $a(m \otimes 1), \Sigma a'(m) \otimes a'' \in \mathrm{Im}.\ell$. Hence $m \otimes a \in \mathrm{Im}.\ell$. Therefore ℓ is an epimorphism. ℓ is a monomorphism, since $\hat{M} \otimes \mathcal{A}$ and $M \otimes \mathcal{A}$ are both vector space and one can count the basis. Therefore

$$\ell : \hat{M} \otimes \mathcal{A} \longrightarrow M \otimes \mathcal{A} \quad \text{is an isomorphism.}$$

 ok

The $\mathcal{A}$ -structure of $M \otimes \mathcal{A}$ depends on M as graded vector space.

(For the other casses, e.g., $M \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$, this is not true.)

$M \otimes \mathcal{A}$ is a right $\mathcal{A}$ -module by

$$(m \otimes a)\bar{a} = m \otimes a\bar{a}$$

Define the right $\mathcal{A}$ -module structure on $\hat{M} \otimes \mathcal{A}$ via ℓ :

$$(m \otimes a)\bar{a} = \ell^{-1}((\ell(m \otimes a))\bar{a}).$$

<u>Theorem</u> This right $\mathcal{A}$ -module structure on $\hat{M} \otimes \mathcal{A}$ is given by the Cartan formula :

$$(m \otimes a)\bar{a} = \Sigma(m)\bar{a}' \otimes a\bar{a}'',$$

where $(m)\bar{a} = \chi(\bar{a})(m)$, χ : the canonical anti-automorphism of the Steenrod algebra.

This is the key lemma.

<u>Proof</u> Consider the diagram :

$$\hat{M} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow[1 \otimes \Psi]{} \hat{M} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow[1 \otimes T \otimes 1]{} \hat{M} \otimes \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$$

$$\xrightarrow[\hat{\Phi} \otimes \Phi]{} \hat{M} \otimes \mathcal{A} \xrightarrow[\ell]{} M \otimes \mathcal{A} .$$

By chasing this diagram we have

$$m \otimes a \otimes b \longrightarrow m \otimes a \otimes b' \otimes b'' \longrightarrow m \otimes b' \otimes a \otimes b''$$

$$\longrightarrow \chi(b')(m) \otimes ab'' \longrightarrow ab''(\chi(b')m \otimes 1)$$

$$= a((b'')'\chi(b')(m) \otimes (b'')'')$$

$$= a((b')'\chi((b')'')(m) \otimes b'')$$

$$= a(m \otimes b)$$

$$= a'm \otimes a''b.$$

Next consider the other diagram :

$$\hat{M} \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow[\ell \otimes 1]{} M \otimes \mathcal{A} \otimes \mathcal{A} \xrightarrow[1 \otimes \Phi]{} M \otimes \mathcal{A}.$$

Similarly we have

$$m \otimes a \otimes b \longrightarrow a(m \otimes 1) \otimes b = a'm \otimes a'' \otimes b \longrightarrow a'(m) \otimes a''b.$$

From this we get the following.

<u>Theorem</u> Let M be an $\mathcal{A}$-module and N be a fixed $\mathcal{B}$-module, where $\mathcal{B}$ is a Hopf subalgebra of $\mathcal{A}$. Then $M \otimes (\mathcal{A} \otimes_{\mathcal{B}} N)$ depends as an $\mathcal{A}$-module only on the $\mathcal{B}$-module structure of M.

If $f : M_1 \longrightarrow M_2$ is an isomorphism as $\mathcal{B}$-module, then the followings are isomorphisms as $\mathcal{A}$-modules

of

$$M_1 \otimes (\mathcal{A} \otimes_{\mathcal{B}} N) \longrightarrow (M_1 \otimes \mathcal{A}) \otimes_{\mathcal{B}} N \xrightarrow{\ell^{-1}\otimes 1} (\hat{M}_1 \otimes \mathcal{A}) \otimes_{\mathcal{B}} N$$

$$\xrightarrow{f\otimes 1\otimes 1} (\hat{M}_2 \otimes \mathcal{A}) \otimes_{\mathcal{B}} N \xrightarrow{\ell\otimes 1} (M_2 \otimes \mathcal{A}) \otimes_{\mathcal{B}} N \longrightarrow M_2 \otimes (\mathcal{A} \otimes_{\mathcal{B}} N).$$

Theorem If M and N are $\mathcal{B}$-modules, then

$$(\hat{M} \otimes \mathcal{A}) \otimes_{\mathcal{B}} N \cong \mathcal{A} \otimes (M \otimes_{\mathcal{B}} N) \text{ as } \mathcal{A}\text{-modules.} \cong M \otimes (\mathcal{A} \otimes_{\mathcal{B}} N)$$

Proof $m \otimes a \otimes n \longrightarrow a \otimes m \otimes n.$

We have to show that $m \otimes a \otimes bn$ and $mb' \otimes ab'' \otimes n$ have the same images under this map.

We have that

$$mb' \otimes ab'' \otimes n \longrightarrow ab'' \otimes \chi(b')m \otimes n$$

$$= a \otimes (b'')'\chi(b')m \otimes (b'')''n$$

$$= a \otimes m \otimes bn.$$

$$m \otimes a \otimes bn \longrightarrow a \otimes m \otimes bn.$$

Let us write the corollaries.

Corollary Let M be a left $\mathcal{A}$-module and N a left $\mathcal{B}$-module.

Let $M \supset \cdots\cdots \supset M^{[i]} \supset \cdots\cdots$ be a filtration of N as $\mathcal{B}$-module.

Then an $\mathcal{A}$-filtration of $M \otimes (\mathcal{A} \underset{\mathcal{B}}{\otimes} N)$ is give by $\mathcal{A} \underset{\mathcal{B}}{\otimes} (M^{[i]} \otimes N)$

with quotients isomorphic as $\mathcal{A}$-modules to $\mathcal{A} \underset{\mathcal{B}}{\otimes} (M^{[i]}/M^{[i-1]} \otimes N)$.

Let us write the corollaries in our applications. $\mathcal{B} = \mathcal{A}_1$,

$N = Z_2$ or $\mathcal{A}_1/\mathcal{A}_1(Sq^3)$

<u>Theorem</u> Assume $M \underset{\mathcal{A}_1}{\cong} \sum_i \mathcal{A}_1/\mathcal{A}_1(Ji)$, $Ji \subset \overline{\mathcal{A}}_1$.

Then $M \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \underset{\mathcal{A}}{\cong} \sum_i \mathcal{A}/\mathcal{A}(Ji)$.

<u>Theorem</u> Assume $M \underset{\mathcal{A}_1}{\cong} \sum_i \mathcal{A}_1/\mathcal{A}_1(Ji)$, $Ji \subset \overline{\mathcal{A}}_1$.

Then $M \otimes \mathcal{A}/\mathcal{A}(Sq^3) \cong$ sum of cyclic $\mathcal{A}$-modules, if no Ji are the following :

$\{Sq^2, Sq^2Sq^1\}$, $\{Sq^3, Sq^2Sq^1\}$, $\{Sq^2Sq^1\}$, $\{Sq^2Sq^1, Sq^5 + Sq^4Sq^1\}$,

$\{Sq^3Sq^1, Sq^5 + Sq^4Sq^1\}$, $\{Sq^5 + Sq^4Sq^1\}$.

Let me give a corollary of this theorem.

<u>Corollauy</u> $BSO \wedge$ <u>MSpin</u> has property P.

$\underline{\text{Proof}}$ $H^*(BSO) \underset{\mathcal{Q}_1}{\cong} \Sigma\, \mathcal{Q}_1/\mathcal{Q}_1(Sq^3) \oplus \Sigma\, \mathcal{Q}_1 \oplus \Sigma\, Z_2,$

where $\mathcal{Q}_1/\mathcal{Q}_1(Sq^3)$, $\mathcal{Q}_1$ and Z_2 correspond to $J = Sq^3$, $J = \emptyset$ and $J = \overline{\mathcal{Q}}_1$ respectively.

Therefore we have

$H^*(BSO) \otimes H^*(\underline{MSpin}) \underset{\mathcal{Q}}{\cong}$ sum of cyclic $\mathcal{Q}$-modules.

We have $E_2 = E_\infty$ in Adams spectral sequence by inspection.

Another important example is $M = H^*(RP^\infty)$. We will describe $\bar{H}^*(RP^\infty) \otimes \mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2)$, $\bar{H}^*(RP^\infty) \otimes \mathcal{Q}/\mathcal{Q}(Sq^3)$ and $\bar{H}^*(RP^\infty) \otimes \mathcal{Q}$,

because this gives $\bar{\Omega}_*^{Spin}(RP^\infty) \cong \Omega_*^{Pin}$.

§ 7. The Pin cobordism.

Spin is a universal covering group of SO. Pin is a universal covering group of O. The component of identity in Pin is Spin.

BPin $\longrightarrow$ BO is constructed by killing w_2. So a manifold has a Pin structure if $w_2(\nu) = 0$, where ν is a normal bundle.

In the Spin case, $w_2(\tau) = 0$ if and only if $w_2(\nu) = 0$, since $w_2(\tau) = w_2(\nu) + w_1(\tau) \cdot w_1(\nu)$.

Note that Ω_*^{Pin} is not a ring, because

$$w_2(\nu_1 \oplus \nu_2) = w_2(\nu_1) + w_1(\nu_1) \cdot w_1(\nu_2) + w_2(\nu_2).$$

But it is a cobordism theory.

Let $G = \text{Pin}$. We have the map

$$BO(1) \times BSG(k) \longrightarrow BG(K + 1)$$

This induces the isomorphism on $H^*(\quad : Z_2)$ in dim. $< k$. Taking the Thom space, we obtain the map

$$MO(1) \wedge MSpin(k) \longrightarrow MPin(k + 1),$$

which induces a mod 2 isomorphism.

Note that $MO(1) \sim S(RP^\infty)$. Therefore

$$\tilde{\Omega}_*^{\text{Spin}}(RP^\infty) = \Omega_*^{\text{Pin}} .$$

We will study $\bar{H}^*(RP^\infty) \otimes \mathcal{A} / \mathcal{A}(Sq^1, Sq^2)$, $\bar{H}^*(RP^\infty) \otimes \mathcal{A} / \mathcal{A}(Sq^3)$ and $\bar{H}^*(RP^\infty) \otimes \mathcal{A}$. Let me state the answers first.

Remember

$$H^*(\underline{MSpin}) = (\mathcal{A}/\mathcal{A}(Sq^1, Sq^2) \otimes X) \oplus (\mathcal{A}/\mathcal{A}(Sq^3) \otimes Y) \oplus (\mathcal{A} \otimes Z) .$$

Each term $H^*(RP^\infty) \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$ contributes the following homotopy to Ω_*^{Pin} :

$$\pi_* = \begin{cases} Z_2 & i \equiv 0, 1 \quad (8) \\ 0 & i \equiv 3, 4, 5, 7 \quad (8) \\ Z_8,\ Z_{16},\ Z_{128} \text{ etc.} & i \equiv 2, 6 \quad (8) \end{cases}$$

where

π_*	2^3	2^4	2^7	2^8	2^{11}	2^{12}
i	2	6	10	14	18	22

For example, it turns out that

$\Omega_2^{Pin} = Z_8$, the representative manifold is the Klein bottle.

Each term $H^*(RP^\infty) \otimes \mathcal{A}/\mathcal{A}(Sq^3)$ contributes the following homotopy to Ω_*^{Pin}

$$\pi_* = \begin{cases} Z_2 & i \equiv 1, 2, 5, 7 \quad (8) \\ Z_2 \oplus Z_2 & i \equiv 6 \quad (8) \\ 0 & i \equiv 3 \quad (8) \\ Z_2,\ Z_4,\ Z_{32} \text{ etc} & i \equiv 4, 0 \quad (8) \end{cases}$$

where

π_*	2	2^2	2^5	2^6	2^9	2^{10}
i	0	4	8	12	16	20

For example, $\Omega_{10}^{Pin} \cong Z_{128} \oplus Z_8 \oplus Z_2$ and the representative manifold of Z_8 is $QP^2 \times$ (Klein bottle).

There exist manifolds $M^8 \in \Omega^{Spin}$ and $M^{10} \in \Omega^{Pin}$ such that $M^8 \times S^1 \times S^1$ represents in Z_2 in Ω^{Spin} but $64([M^{10}]) = [M^8 \times S^1 \times S^1]$ in Ω^{Pin}.

Let us state some theorems about Pin cobordism. Let $R^i = \bar{H}^{i+1}(RP^\infty)$ as an $\mathcal{A}$-module.

<u>Proposition</u> As an $\mathcal{A}_1$-module, R has a filtration

$$R \supset \text{-----} \supset R^{[4i+2]} \supset R^{[4i-2]} \supset \text{------} \supset R^{[2]} \supset R^{[0]},$$

where $R^{[i]}$ is an $\mathcal{A}_1$-module generated by R^j, $j \leq i$ and $R^{[4i+2]}/R^{[4i-2]} = \mathcal{A}_1/\mathcal{A}_1(Sq^1)$, $R^{[2]}/R^{[0]} = \mathcal{A}_1/\mathcal{A}_1(Sq^1)$, $R^{[0]} = \mathcal{A}_1/\mathcal{A}_1(Sq^2)$.

Extension is given by $Sq^1(r_{4i+2}) = (Sq^5 + Sq^4Sq^1)(r_{4i-2})$, $Sq^1(r_2) = Sq^2Sq^1(r_0)$.

Proof is straightforward.

So then we have

<u>Theorem</u> $R \otimes \mathcal{A}/\mathcal{A}(Sq^1, Sq^2)$ has a filtration as $\mathcal{A}$-modules

$$\supset \text{-------} \supset F^{(4i+2)} \supset F^{(4i-2)} \supset \text{------} \supset F^{(2)} \supset F^{(0)}$$

with $F^{(4i+2)}/F^{(4i-2)} = \mathcal{A}/\mathcal{A}(Sq^1)$, $F^{(2)}/F^{(0)} = \mathcal{A}/\mathcal{A}(Sq^1)$ and $F^{(0)} = \mathcal{A}/\mathcal{A}(Sq^1)$.

$\uparrow Sq^2$?

<u>Proof</u> Corollary of the previous theorem.

A little more complicated is the other case:

<u>Theorem</u> $R \otimes \mathcal{Q}/\mathcal{Q}(Sq^3)$ has a filtration as $\mathcal{Q}$ -modules

$$\supset \text{--------} \supset G^{(i)} \supset \text{--------},$$

where $G^{(4i+2)}/G^{(4i+1)} = \mathcal{Q}$, $G^{(4i+3)}/G^{(4i+2)} = 0$

$G^{(4i+4)}/G^{(4i+3)} = \mathcal{Q}/\mathcal{Q}(Sq^1)$, $G^{(4i+5)}/G^{(4i+4)} = \mathcal{Q}$

and $G^{(1)}/G^{(0)} = \mathcal{Q}$, $G^{(0)} = \mathcal{Q}/\mathcal{Q}(Sq^5Sq^1)$.

<u>Proof</u> Corollary of the above theorem (One should calculate $\mathcal{Q} \otimes_{\mathcal{Q}_1} (R^{[i]}/R^{[i-1]} \otimes \mathcal{Q}_1/\mathcal{Q}_1(Sq^3)))$.

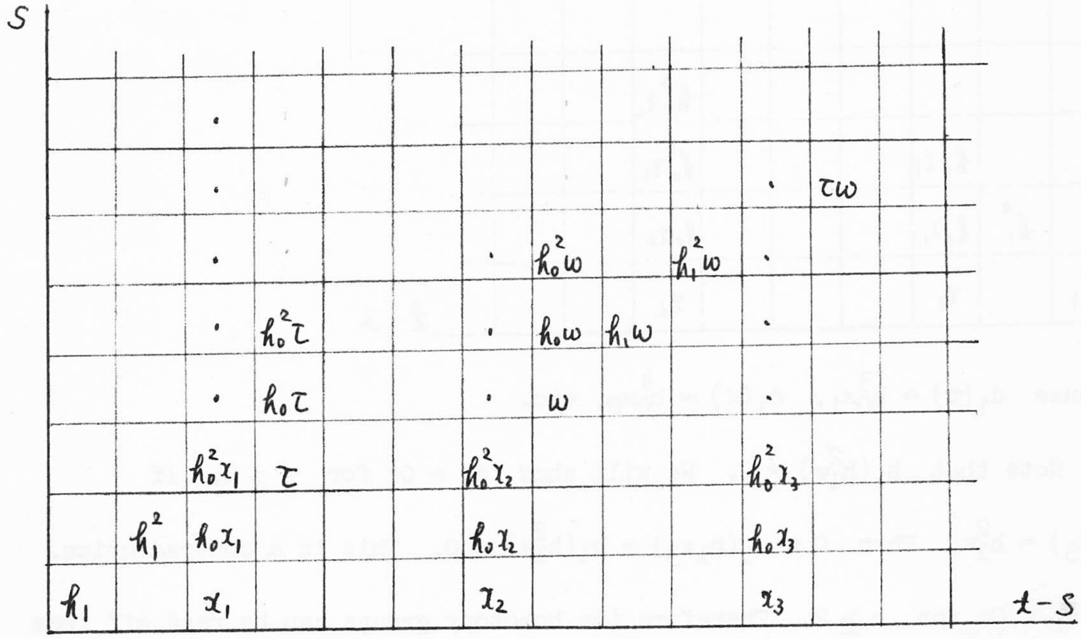
We want to study

$$\mathrm{Ext}_{\mathcal{Q}}(R \otimes \mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2), Z_2)$$

by knowing the filtration of $R \otimes \mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2))$.

Intuitively we assume

$R \otimes \mathcal{Q}/\mathcal{Q}(Sq^1, Sq^2)$ = direct sums of $F^{(4i+2)}/F^{(4i-2)}$.



To obtain the correct E_2 put $d_1: E_1^{a,b} \longrightarrow E_1^{a-1,b+1}$. We need the following theorem of Adams;

Theorem of Adams

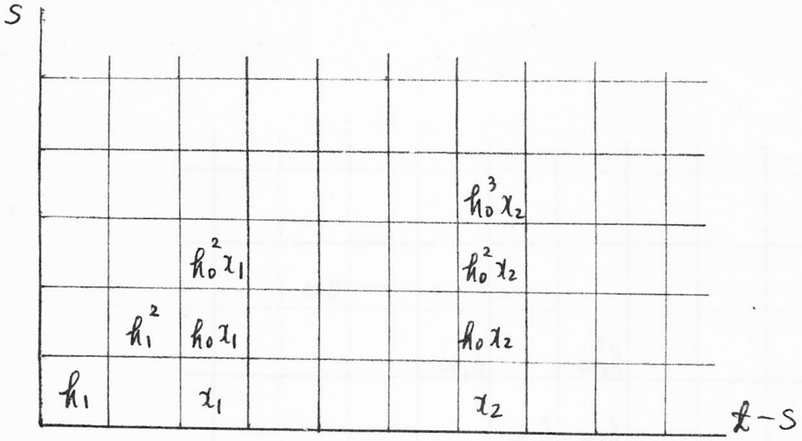
If $H(M, Q_0) = 0$, then there are no elements of ∞-height in $\mathrm{Ext}_{\mathcal{A}}(M, Z_2)$.

(This is not difficult to prove)

Note:

$H(R \otimes \mathcal{A} / \mathcal{A}(Sq^1, Sq^2), Q_0) = H(R, Q_0) \otimes H(\mathcal{A} / \mathcal{A}(Sq^1, Sq^2), Q_0)$,

where $H(R, Q_0) = 0$. Hence the E_2-term is

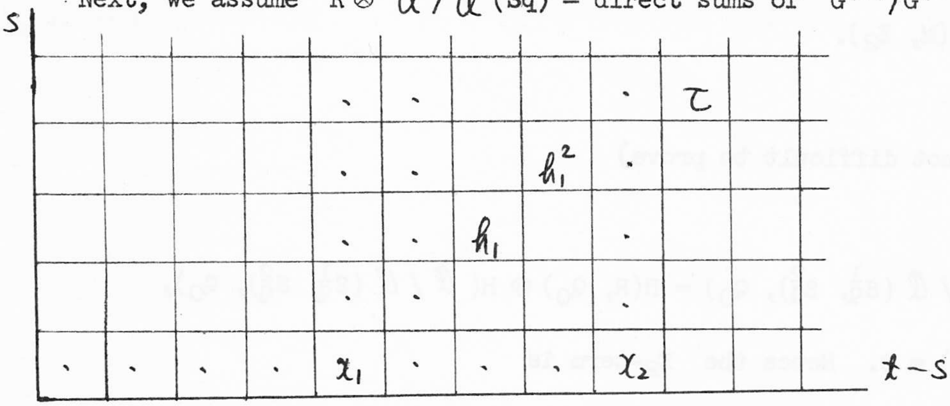


because $d_1(\tau) = h_0^3x_1$, $d_1(w) = h_0^4x_2$, etc.

Note that $h_1(h_1^2w) \neq 0$. We will show $d_r = 0$ for $r \geq 2$. If $d_5(x_3) = h_1^2w$, Then $0 = d_5(h_1x_3) = h_1(h_1^2w) \neq 0$. This is a contradiction. So $d_r = 0$ for $r \geq 2$. Therefore the homotopy groups can be read off from the table.

$$\pi_i = \begin{cases} Z_2 & i \equiv 0,\ 1 \quad (8) \\ 0 & i \equiv 3,\ 4,\ 5,\ 7 \quad (8) \\ Z_8,\ Z_{16},\ Z_{128} \text{ etc.} & i \equiv 2,\ 6 \quad (8) \end{cases}$$

Next, we assume $R \otimes \mathcal{A}/\mathcal{A}(Sq^3)$ = direct sums of $G^{(i)}/G^{(i-1)}$.



d_1 is similar to the above. Note that $E_2 = E_\infty$ in the Adams spectral sequence. Therefore MPin has property P. So we have

Theorem $\operatorname{Im}(\Omega_*^{Pin} \longrightarrow \mathcal{N}_*)$ = all cobordism classes all of whose Stiefel-Whitney numbers involving $w_2(\nu)$ vanish.

§ 8. The $Spin^C$-cobordism.

Let me now state some results about $Spin^C$-cobordism.

$Spin^C$ = complex spin group.

$BSpin^C \longrightarrow BSO$ is obtained by killing w_3, that is,

$$\begin{array}{ccc} BSpin^C & & \text{path space} \\ \downarrow & & \downarrow \\ BSO & \xrightarrow{\delta^*(w_2)} & K(Z, 3) \end{array}$$

where $\delta^*(w_2)$ is the image of the Bockstein operator of w_2.

$Spin^C$ is a natural theory for K-theory because a bundle is orientable with respect to K-theory $\Longleftrightarrow$ the bundle has a $Spin^C$-structure.

The methods for calculating MSpin work for $MSpin^C$ and are much easier. Let me state the answers. They are

Theorem

$$H^*(MSpin^C) = (\mathcal{A}/\mathcal{A}(Q_0, Q_1) \otimes X) \oplus (\mathcal{A} \otimes Z)$$

Theorem Let $[M] \in \Omega_*^{Spin^C}$, then $[M] = 0$

$\Longleftrightarrow$ all mod 2 and all integral characteristic numbers vanish.

(One needs no K-theory)

<u>Theorem</u> $\mathrm{Im}(\Omega_*^{\mathrm{Spin}^c} \longrightarrow \mathfrak{N}_*) =$ all cobordism classes all of whose Stiefel-Whitney numbers involving ω_1 and ω_3 vanish.

One might

<u>Conjecture</u> : $\Omega_*^{\mathrm{Spin}^c}$ is generated as a ring by $\mathrm{Im}(\Omega_*^{\mathrm{Spin}} \longrightarrow \Omega_*^{\mathrm{Spin}^c})$ and $\mathrm{Im}(\Omega_*^{U} \longrightarrow \Omega_*^{\mathrm{Spin}^c})$.

This is true in dim. ≤ 30 but it is false in dim. 31.

$$\begin{tikzcd} \mathrm{Spin} \arrow[r] & \mathrm{Spin}^c \\ U \arrow[ur] & \end{tikzcd}$$

One could consider Pin^c and the same methods again work well.

For some pages let p be odd. Let me discuss the structure of BSO and BU ignoring all primes but p. The main theorem is that BSO is decomposable in the classical sense. For this we develop some machinery. Let B_p be a space like BSO with

$$\pi_i(B_p) = \begin{cases} 0 & i \not\equiv 0 \quad (4) \\ Z & i \equiv 0 \quad (4) \end{cases}$$

and all k-invariants of order power of p.

First theorem is

<u>Theorem</u> Let K be a space such that

$$\pi_i(K) = \begin{cases} 0 & i \not\equiv 0 \quad (4) \\ Z & i \equiv 0 \quad (4) \end{cases} \qquad \text{mod } \mathcal{C}_p$$

and $H^{4i+1}(K:Z) \in \mathcal{C}_p$. Then there exists a map $f:K \longrightarrow K_p$ which is mod p homotopy equivalence i.e., f^* is isomorphism on $H^*(\quad :Z_p)$.

<u>Proof</u> Given a space K we form the Postnikov system

$$\begin{tikzcd}
K \arrow[d, dashed] & & \\
K^{(i)} \arrow[d] & & \text{path space} \arrow[d] \\
K^{(i-1)} \arrow[rr] \arrow[d, dashed] & & K(\pi_i(K), i+1) \\
\text{point} & &
\end{tikzcd}$$

with k-invariants $k^{(i+1)}(K) \in H^{i+1}(K^{(i-1)}:\pi_i(K))$. These k-invariants determine the fibrations.

Consider the diagram

$$\begin{tikzcd}
 & B_p \arrow[d, dashed] & \\
 & B_p^{(4t)} \arrow[d] & \\
 & B_p^{(4t-4)} \arrow[r] \arrow[d, dashed] & K(Z, 4t+1) \\
f_1:K \arrow[r] \arrow[uuur, dashed] & K(Z, 4) &
\end{tikzcd}$$

Inductively we lift the map f_1.

Assume we have $f_{t-1}:K \longrightarrow B_p^{(4t-4)}$

$$\begin{tikzcd}
 & B_p^{(4t)} \arrow[d] \\
K \arrow[ur, "f_t"] \arrow[r, "f_{t-1}"'] & B^{(4t-4)}
\end{tikzcd}$$

The obstruction to finding f_t is $f_{t-1}^*(k^{4t+1}(B_p)) \in H^{4t+1}(K:Z)$. Since $k^{4t+1}(B_p)$ is of order power of p, $f_{t-1}^*(k^{4t+1}(B_p)) = 0$.

We set $f = f_\infty:K \longrightarrow B_p$. We must, however, show that $f^{(4t)*}$ is isomorphism on $H^*(\quad :Z_p)$ for $f^{(4t)}:K^{(4t)} \longrightarrow B_p^{(4t)}$. If we do this, f^* is also an isomorphism on $H^*(\quad :Z_p)$.

We have the following diagram:

$$\begin{tikzcd}
K(Z, 4t) \arrow[r, "g"] \arrow[d] & K(Z, 4t) \arrow[d] & \\
K^{(4t)} \arrow[r, "f^{(4t)}"] \arrow[d] & B_p^{(4t)} \arrow[d] & \\
K^{(4t-4)} \arrow[r, "f^{(4t-4)}"] & B_p^{(4t-4)} \arrow[r] & K(Z, 4t+1)
\end{tikzcd}$$

We assume $f^{(4t-4)*}$ is an isomorphism on $H^*(\quad :Z_p)$. So we have

$$H^{4t+1}(B_p^{(4t-4)}:Z) \cong Z_p\phi(t) \text{ with a generator } x = k^{4t+1}(B_p).$$

Therefore

$$H^{4t+1}(K^{(4t-4)}:Z) \cong Z_p\phi(t) \text{ which is mapped by } f^{(4t-4)*}.$$

The k-invariant of K is $sx = k^{4t+1}(K)$

Then we have

$$s \not\equiv 0 \quad (p) \quad \text{or} \quad H^{4t+1}(K, Z) \notin \mathcal{C}_p.$$

which implies $s \not\equiv 0 \quad (p)$.

For the generator $\iota \in H^{4t}(Z, 4t)$ we have

$$g^*(\iota) = a\iota.$$

By naturality $x = a\,s\,x$. So $a \not\equiv 0 \quad (p)$. Therefore g^* is isomorphism on $H^*(\quad :Z_p)$. Hence $f^{(4t)*}$ is isomorphism on $H^*(\quad :Z_p)$. This finishes the induction.

This argument works for $x \neq 0$.

If $x = 0$, that is $Z_p\phi(t) = 0$, then $B_p^{(4t)} = B_p^{(4t-4)} \times K(Z, 4t)$, and we should change $f^{(4t)}$ and extend to new f. Q.E.D.

Let $K \longrightarrow K^{(i-1)}$ be a fibration with a fibre $K_{(i)}$ such that $\pi_j(K_{(i)}) = 0$ for $j < i$.

The better and more useful theorem is the following

<u>Theorem</u> Let K be a space such that

$$\pi_i(K) = \begin{cases} 0 & i \not\equiv 0 \quad (4) \\ Z & i \equiv 0 \quad (4) \end{cases} \qquad \text{mod } \mathcal{C}_p$$

and the first k-invariant of $K_{(4t)}$ in $H^{4t+2p-1}(K_{(4t)}^{(4t+2p-6)} : Z) \cong Z_p$ is $\lambda\beta\mathcal{P}^1$, $\lambda \not\equiv 0$ (p). Then there exists a map

$$f : K \longrightarrow B_p$$

which is a mod p homotopy equivalence.

<u>Proof</u>

$$\begin{tikzcd}
K \arrow[dd, dashed] \arrow[dr, dashed, "f_t"] \arrow[ddr, "f_{t-1}"] & B_p \\
 & B_p^{(4t)} \arrow[d] \\
K^{(4t-4)} \arrow[d] & B_p^{(4t-4)} \arrow[d, dashed] \\
K^{(2p-2)} \arrow[d, dashed] & K(Z, 4) \times K(Z, 8) \times \cdots \times K(Z, 2p-2) \arrow[d, dashed] \\
K(Z, 4) & K(Z, 4)
\end{tikzcd}$$

(Inductive hypothesis) Assume f_{t-1} exist such that $f_{t-1}^{(4t-4)}:K^{(4t-4)}$ $\longrightarrow B_p^{(4t-4)}$ is an isomorphism on $H^*(\ \ :Z_p)$. Therefore $H^{4t+1}(K^{(4t-4)}:Z)$ $Z_p\phi(t)$ with a generator x.

We will prove that the k-invariant $k^{4t+1}(K) = sx$ with $s \not\equiv 0\ (p)$. For, if $s \equiv 0\ (p)$, then consider the map $K^{(4t-4)}_{(4t-2p+2)} \longrightarrow K^{(4t-4)}$ inducing the homomorphism $Z_p\phi(t) \longrightarrow Z_p$ which maps sx to a non-zero element. Hence $s \not\equiv 0\ (p)$. Therefore we obtain $H^{4t+1}(K:Z) \in \mathcal{C}_p$. Now we follow the same proof as of the previous theorem.

<u>Theorem</u> There exists a space Y_p such that

$$\pi_i(Y_p) = \begin{cases} 0 & i \not\equiv 0 \quad (2p-2) \\ Z & i \equiv 0 \quad (2p-2) \end{cases}$$

and the first k-invariant in $H^{4t+2p-1}(Y^{(4t)}_{p(4t)}:Z) \cong Z_p$ is $\lambda\beta\mathcal{P}^1$, $\lambda \not\equiv 0\ (p)$.

This is proved next time. Assume this for the moment, then we have

<u>Corollary</u>

$$BSO \underset{p}{\sim} \prod_{i=0}^{\frac{p-1}{2}-1} \Omega^{4i}Y_p.$$

$$BU \underset{p}{\sim} \prod_{i=0}^{p-2} \Omega^{2i}Y_p.$$

These are mod p H-space equivalences.

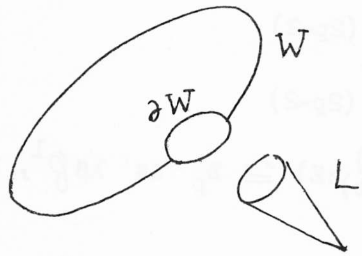
This is seen by inspection. This theorem is useful for some calculations of BSPL.

§ 9. The cobordism with singularities.

Let me start today by describing "Cobordism with singularities". This is a theory of D. Sullivan.

We start with Ω_*^U. Let $\mathbb{C} = [\mathbb{C}] \in \Omega_c^U$. We fix $\mathbb{C}$ for a while. Consider a manifold W^n such that $\partial W^n \approx L \times \mathbb{C}$. We form

$$\bar{W} = W \cup L \times \text{cone } \mathbb{C} \text{ along boundary.}$$



These are "closed manifolds" of new theory. The bounding manifolds in new theory are W^{n+1} such that $\partial W^{n+1} \approx L \times \mathbb{C} \cup A$ along $\partial L \times \mathbb{C}$ (We also have an identification $\partial A = \partial L \times \mathbb{C}$).

Sullivan proves that one can form a bordism theory $\Omega_*^{\mathbb{C}}(K)$ which is a generalized homology theory. One can relate the coefficient groups:

$$\cdots \longrightarrow \Omega_{n-c}^U \xrightarrow{\times \mathbb{C}} \Omega_n^U \longrightarrow \Omega_n^{\mathbb{C}} \longrightarrow \Omega_{n-c-1}^U \xrightarrow{\times \mathbb{C}} \Omega_{n-1}^U \longrightarrow \cdots .$$

It is easy to check that this is an exact sequence. We know the ring $\Omega_*^U = Z[c_1, \ldots\ldots]$, $c_i \in \Omega_{2i}^U$. So, if $\mathbb{C} \neq 0$, then multiplication $\times \mathbb{C}$ is a monomorphism, that is, we have

$$0 \longrightarrow \Omega_{n-c}^U \xrightarrow{\times \mathbb{C}} \Omega_n^U \longrightarrow \Omega_n \longrightarrow 0,$$

whence $\Omega_n^{\mathbb{C}} = Z[c_1, \ldots\ldots] / (\mathbb{C})$.

Repeating this process on $\Omega_n^{\mathbb{C}}$, fixing $d\!l \in \Omega_n^{\mathbb{C}}$, one obtains another exact sequence:

$$\cdots \longrightarrow \Omega_{n-d}^{\mathbb{C}} \xrightarrow{\times d\!l} \Omega_n^{\mathbb{C}} \longrightarrow \Omega_n^{\mathbb{C}, d\!l} \longrightarrow \Omega_{n-d-1}^{\mathbb{C}} \longrightarrow \cdots .$$

If we choose $x_1, x_2, \ldots$ such that x_{i+1} is not zero divisor of $\Omega^U/(x_1, \ldots, x_i)$, then

$$\Omega_*^{x_1, \ldots, x_{i+1}} = \Omega^U/(x_1, \ldots, x_{i+1}).$$

Here again one obtains a generalized homology theory.

<u>Example 1</u> $\mathbb{C} = n$ points. Then one obtains $\Omega_*^U \otimes Z_n$.

<u>Example 2</u> $x_1, \ldots, x_i, \ldots = c_1, \ldots, c_i, \ldots$, then one obtains $H_*(\quad :Z)$ the ordinary homology theory, because the coefficients are $Z[c_1, \ldots] / (c_1, \ldots\ldots) \cong Z$.

<u>Example 3</u> $x_1, x_2, \ldots = c_2, c_3, \ldots$. (first choose generators c_i such that Todd genus $T(c_i) = 0$ if $i > 1$). i.e., you kill off c_i except c_1. Then one obtains K-theory $K_*(pt)$. Note $K_*(pt) = Z[c_1]$.

<u>Example 4</u> Choose $x_1, \ldots = c_1, c_3, c_4, c_5, \ldots$ (leaving out c_2) generators c_{2i} chosen such that index $I(c_{2i}) = 0$ $(c_2 = CP^2)$, then one obtains a theory $V_*(\quad)$. Now $V_*(\quad) = \pi_*(\underline{V}) = Z[c_2]$, where $\underline{V}$ is a spectrum. Assume $\underline{V}$ is an Ω-spectrum, $\Omega V_{i+1} = V_i$, then $\pi_*(V_0) = Z[c_2]$ (cf. Brown or Whitehead's paper). Using surgery, one can prove $V_0 \sim F/PL$ for all primes except 2.

<u>Example 5</u> Choose $x_1, \ldots = c_1, \ldots, \hat{c}_{p-1}, \ldots$, then one obtains $V_*^{\prime}(pt) = Z[c_{p-1}]$, where $\dim c_{p-1} = 2p - 2$ and p is an odd prime.

Let $Y_p = V'_0$. $\pi_*(Y_p) = Z\,[c_{p-1}]$. I want to claim that $\underline{V}'$ is periodic of period $2p - 2$, roughly speaking

$$\Omega^{2p-2}\,\underline{V}' \sim \underline{V}'.$$

We have a map

$$S^{2p-2} \wedge \underline{V}' \longrightarrow \underline{V}' \wedge \underline{V}' \longrightarrow \underline{V}',$$

and hence the associate map

$$\underline{V}' \longrightarrow \Omega^{2p-2}\,\underline{V}'$$

Considering the induced homomorphism on π_*, this sends $(c_{p-1})^t$ to $(c_{p-1})^{t+1}$. Therefore it is an isomorphism on π_*, because $\pi_*(Y_p)$ is a polynomial ring on one generator.

Finally note that the first k-invariant of $\underline{V}'$ is not zero. Proof is to compare with spectrum $\underline{MU} \longrightarrow \underline{V}'$. (We know the first k-invariant of $\underline{MU}$ and by naturality one can check it).

<u>Theorem</u> There exists a space Y_p such that

$$\pi_i(Y_p) = \begin{cases} Z & i \equiv 0 \quad (2p-2) \\ 0 & i \not\equiv 0 \quad (2p-2) \end{cases}$$

and the first k-invariant of $Y_p(i(2p-2))$ is nonzero.

Proof is by the construction of example 5.

<u>Corollary</u>

$$F/Pl \underset{3}{\sim} BSO.$$

Proof $F/PL \underset{p}{\sim} V_0 = Y_3 \underset{3}{\sim} BSO$,

where p is any odd prime.

I state the following theorem without proof.

Theorem of Sullivan

$$F/PL \sim BSO \text{ for any odd prime.}$$

It seems reasonable to construct Y_p directly.

§ 10. The PL-cobordism.

Now we discuss PL-cobordism. There is an important theory of Williamson:

$$\mathcal{N}_n^{PL} \cong \varinjlim \pi_{n+i}(MPL_i) = \pi_n(\underline{MPL}).$$

So the question is how to compute this. There is a classifying space BPL_i and some limiting process $BPL_i \longrightarrow BPL$. Moreover we have a diagram

$$\begin{array}{ccc} BPL_i & \longrightarrow & BPL \\ \uparrow & & \uparrow \\ BO_i & \longrightarrow & BO \end{array}$$

So we have the homomorphism

$$\theta : H^*(BPL:Z_2) \longrightarrow H^*(BO:Z_2) = Z_2[w_1, \ldots\ldots].$$

By definition of w_i, w_i^{PL} can be defined in $H^i(BPL:Z_2)$ such that

$$\theta(w_i^{PL}) = w_i.$$

Define $\phi: H^*(BO:Z_2) \longrightarrow H^*(BPL:Z_2)$ by $\phi(w_i) = w_i^{PL}$, then ϕ is a map of algebras. One obtains

$$\psi(w_i^{PL}) = \Sigma\, w_i^{PL} \otimes w_{i-j}^{PL}$$

by the usual proof. Therefore ϕ is a map of Hopf algebras.

Recall the definition of $w_i^{PL}: w_i^{PL} = \varphi^{-1} Sq^i(U)$.

The question is if the equality

$$Sq^i(w_j^{PL}) = \Sigma\, (\quad) w^{PL} \cdot w^{PL}$$

hold. Using the Cartan formula, the Adem relations and induction, one can prove

$$Sq^i(w_j^{PL}) = \text{some polynomial in } w_i^{PL}\text{'s}.$$

Therefore it is equal to the correct polynomial, because under θ it goes into the correct polynomial.

<u>Lemma</u> ϕ is a map of Hopf algebra over $\mathcal{A}$. Define

$$C = H^*(BPL)/\phi(\bar{H}^*(BO) \cdot H^*(BPL)),$$

where $\bar{H}^*$ means the elements of positive degree. Then C is a Hopf algebra over $\mathcal{A}$.

Applying Milnor-Moore theory one gets

<u>Theorem</u> The composition

$$H^*(BPL) \xrightarrow{\psi} H^*(BPL) \otimes H^*(BPL) \xrightarrow{\theta \otimes \pi} H^*(BPL) \otimes C,$$

where π is projection, gives an isomorphism of Hopf algebra over $\mathcal{a}$.

<u>Theorem</u> As an algebra,

$$\mathcal{N}_*^{PL} \cong \mathcal{N}_* \otimes C^*,$$

where C is a Hopf algebra as preceding theorem and C^* is a dual of C.

Remember that

$H^*(\underline{BG}) \xrightarrow{\Phi} H^*(\underline{MG})$ is an isomorphism of coalgebras for $G = O$ and PL. One can define a right operation on $H^*(BO)$ by

$$(h)a = \Phi^{-1}\chi(a)(\Phi(h)).$$

We have that $h: \mathcal{N}_* \longrightarrow H_*(\underline{MO})$ is a monomorphism and that $h^*: H^*(\underline{MO}) \longrightarrow (\mathcal{N}_*)^*$ is an epimorphism with kernel $\bar{\mathcal{a}} \cdot H^*(MO)$.

Using the Thom isomorphism, one gets that

$$H^*(\underline{BO}) \longrightarrow H^*(\underline{MO}) \longrightarrow (\mathcal{N}_*)^*$$

is an epimorphism with kernel $H^*(BO) \cdot \bar{\mathcal{a}}$ and this is a map of coalgebras.

I want to consider those $S_w(W) = S_w$ such that w has no members of the form $2^i - 1$. Let S = vector space spanned by such elements in $H^*(BO)$.

<u>Lemma</u> $S \longrightarrow (\mathcal{N}_*)^*$ is an isomorphism of coalgebras.

Proof The isomorphism is given by Thom. We have

$$\psi(S_w) = \sum_{w_1 w_2 = w} S_{w_1} \otimes S_{w_2},$$

and note that w_1 and w_2 are of the above type. Therefore S is closed under the diagonal map.

The composition

$$(\mathcal{N}_*)^* \otimes C \longrightarrow S \otimes C \longrightarrow H^*(BO) \otimes C = H^*(BPL) \longrightarrow (\mathcal{N}^{PL}_*)^*$$

is a map of coalgebras and one can check that it is an isomorphism as vector space.

So, dually, $\mathcal{N}^{PL}_* \cong \mathcal{N}_* \otimes C^*$ as algebra. This has some corollaries.

Corollary If $M^n \not\sim C^\infty$-manifold and N is a C^∞-manifold, $N \not\sim 0$, then $M \times N \not\sim C^\infty$-manifold. The following results are known on the structure of C.

Theorem $C_i = 0$ for $i < 8$.

$C_8 = Z_2$.

$C_9 = Z_2 \oplus Z_2$.

$C_i \neq 0$ for $i \geq 24$.

One is also interested in the orientable case Ω^{SPL}_*.

The same methods prove that

$$H^*(BSPL) \cong H^*(BSO) \otimes C$$

with the same C as unoriented case.

And the same proof shows that

$$H^*(\underline{M}SPL) \cong H^*(MSO) \otimes C \text{ as coalgebra.}$$

From this one can prove that

$$H^*(\underline{M}SPL) = \Sigma\, \mathcal{A}/\mathcal{A}(Sq^1) \oplus \text{free } \mathcal{A}\text{-module as } \mathcal{A}\text{-module.}$$

<u>Technical lemma</u>

If $\underline{M}$ is a spectrum with

$$H^*(\underline{M}) \cong \Sigma\, \mathcal{A}/\mathcal{A}(Sq^1) \otimes \Sigma\, \mathcal{A}.$$

then $\underline{M} \underset{2}{\sim} \vee\, \underline{K}(Z,\ldots) \vee \underline{K}(Z_{2^r},\ldots)$.

(Note: $H^*(\underline{K}(Z_8, 0)) = \mathcal{A}/\mathcal{A}(Sq^1) \oplus \mathcal{A}/\mathcal{A}(Sq^1)$)

This means that in Ω_*^{SPL} for $p = 2$ every manifold can be detected with characteristic classes with coefficients in Z and Z_{2^r}.

For p:odd, what is the structure of $H^*(BSPL:Z_p)$?

Using $H^*(BSF:Z_p) \cong Z_p[q_i] \otimes E(\beta q_i) \otimes C$ (proved recently upstairs) and direct computation, one can prove

$$H^*(BSPL:Z_p) \cong H^*(BSO:Z_p) \otimes C$$

in dimensions $< (p^2 + p + 1)(2p - 2) - 1$.

Therefore one can try to compute $H^*(\underline{M}SPL:Z_p)$ as modules over

Here C is known explicitly up to $2p(2p-2)$.

Some pages later we see, for example, that

$$H^*(\underline{M}SPL:Z_3) = \Sigma\, \mathcal{Q}/(\beta) \oplus \text{free in } \dim < 27,$$

where

$$\mathcal{Q}/(\beta) = \mathcal{Q}/\mathcal{Q}(Q_0, Q_1, Q_2, \ldots).$$

The part $\mathcal{Q}/(\beta)$ comes from Ω_*^{SO} and the free part comes from PL - , but not C^∞-manifolds.

Note that CP^2, CP^4, CP^6..... are generators and new things are

11	Z_3	
19	Z_3	
22	Z_3	
23	$Z_3 \oplus Z_3 \oplus Z_3$	
27	Z_9	($H^*(\underline{M}SPL, Z_3)$ is no longer free)

Note, for example, that $M^{11} \times CP^2 = 0$, which is different from $\mathfrak{N}_*^{PL}$.

The End.

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