Journal of Homotopy and Related Structures (2025) 20:511–521
https://doi.org/10.1007/s40062-025-00379-4

The connective *KO*-theory of the Eilenberg–MacLane space $K(\mathbb{Z}_2, 2)$, I: the E_2 page

Donald M. Davis[1] · **W. Stephen Wilson**[2]

Received: 10 October 2024 / Accepted: 15 August 2025 / Published online: 28 August 2025

Abstract

We compute the E_2 page of the Adams spectral sequence converging to the connective KO-theory of the second mod 2 Eilenberg–MacLane space, $ko_*(K(\mathbb{Z}_2, 2))$, where $\mathbb{Z}_2$ is the cyclic group of order 2. This required a careful analysis of the structure of $H^*(K(\mathbb{Z}_2, 2); \mathbb{Z}_2)$ as a module over the subalgebra of the Steenrod algebra generated by Sq^1 and Sq^2. Complete analysis of the spectral sequence is performed in [8].

Keywords Adams spectral sequence · Steenrod algebra · Connective KO-theory · Eilenberg–MacLane space

Mathematics Subject Classification 55S10 · 55T15 · 55N20 · 55N15

1 Introduction

Let $\mathbb{Z}_2 = \mathbb{Z}/2$ and let K_2 denote the Eilenberg–MacLane space $K(\mathbb{Z}_2, 2)$. In [9], the authors gave a complete determination of the connective complex K-theory groups $ku_*(K_2)$ and $ku^*(K_2)$. The original motivation for this work was from [16] and [10], which studied Stiefel–Whitney classes of Spin manifolds. Because of the relationship [2] of the Spin cobordism spectrum and the spectrum ko for connective real K-theory, information about $ko_*(K(\mathbb{Z}_2, n))$ gave useful results about Spin manifolds. For complete calculations the authors were led to the more tractable ku groups. In this paper, we return to the ko groups.

We give a complete determination of the E_2 page of the Adams spectral sequences (ASS) converging to $ko_*(K_2)$ and $ko^*(K_2)$. In [8], we will complete the calculation

---

Communicated by Haynes Miller.

✉ Donald M. Davis
dmd1@lehigh.edu

W. Stephen Wilson
wswilsonmath@gmail.com

[1] Department of Mathematics, Lehigh University, Bethlehem, PA 18015, USA

[2] Department of Mathematics, Johns Hopkins University, Baltimore, MD 01220, USA

by determining the differentials and extensions in the spectral sequences. We choose to split this E_2 work off because we feel that it involves some clever arguments that we would not want to have obscured in a paper with massive ASS charts.

Most of our focus will be on the homology groups $ko_*(K_2)$, in part because of its connection with the motivating problem and in part because its ASS is of a more familiar form than that for $ko^*(-)$. In [9], most of the work was done for the cohomology groups $ku^*(K_2)$, largely because of the product structure. That structure, along with a comparison with the mod-p groups $k(1)^*(K_2)$, enabled us to find the differentials in the spectral sequence for $ku^*(K_2)$, and we can use that information to deduce differentials in the other spectral sequences. Similarly to the situation for ku in [9], the ko-homology and ko-cohomology groups of K_2 are Pontryagin duals of one another. We discuss this in Sect. 4.

Let A_1 denote the subalgebra of the mod-2 Steenrod algebra generated by Sq^1 and Sq^2, and let E_1 denote the exterior subalgebra generated by the Milnor primitives $Q_0 = \mathrm{Sq}^1$ and $Q_1 = \mathrm{Sq}^1\,\mathrm{Sq}^2 + \mathrm{Sq}^2\,\mathrm{Sq}^1$. The ASS converging to $ko_*(X)$ has $E_2^{s,t} = \mathrm{Ext}_{A_1}^{s,t}(H^*X, \mathbb{Z}_2)$, while that for $ku_*(X)$ has $E_2 = \mathrm{Ext}_{E_1}(H^*X, \mathbb{Z}_2)$. All cohomology groups have coefficients in $\mathbb{Z}_2$. The first step toward $ku^*(K_2)$ was finding a splitting of H^*K_2 as a direct sum of reduced E_1-modules ([9, Proposition 2.11 and (2.16)]). (A *reduced* module is one containing no free submodules.) In Section 3, we describe a corresponding splitting as A_1-modules (Theorem 3.8) and the groups $\mathrm{Ext}_{A_1}(-, \mathbb{Z}_2)$ for all of the summands. This then will be the E_2 page of the ASS, the main result of this paper.

We thank the referee for a careful reading and several useful suggestions.

2 The A_1-summands M_k

An important part of the E_1 splitting of H^*K_2 was a family of E_1-modules M_k for $k \geq 4$ ([9, (2.13), (2.14), (2.15)]). In this section, we find corresponding A_1-modules, which we also call M_k. Although the structure of these A_1-modules as E_1-modules is very similar to that of the corresponding E_1-modules of [9] (in fact isomorphic if $k \equiv 0, 1 \bmod 4$), finding classes with the correct Sq^2 behavior was a nontrivial task.

Let u_0 denote the nonzero element of $H^2(K_2)$, and define u_j inductively by $u_{j+1} = \mathrm{Sq}^{2^j} u_j$. Then $H^*(K_2) = \mathbb{Z}_2[u_j : j \geq 0]$ with $|u_j| = 2^j + 1$.([14]) Let $S = (\mathrm{Sq}^1, \mathrm{Sq}^2)$. One easily checks that

$$S(u_j) = \begin{cases} (u_1, u_0^2) & j = 0 \\ (0, u_2) & j = 1 \\ (u_{j-1}^2, 0) & j \geq 2. \end{cases}$$

In Lemma 2.1 we replace u_j with generators x_j for $j \geq 4$ with similar properties except that $\mathrm{Sq}^2\,\mathrm{Sq}^1(x_4) = 0$.

Lemma 2.1 *There are elements $x_j \in H^{2^j+1}(K_2)$ for $j \geq 4$ satisfying*

(1) $x_j \equiv u_j$ mod decomposables,

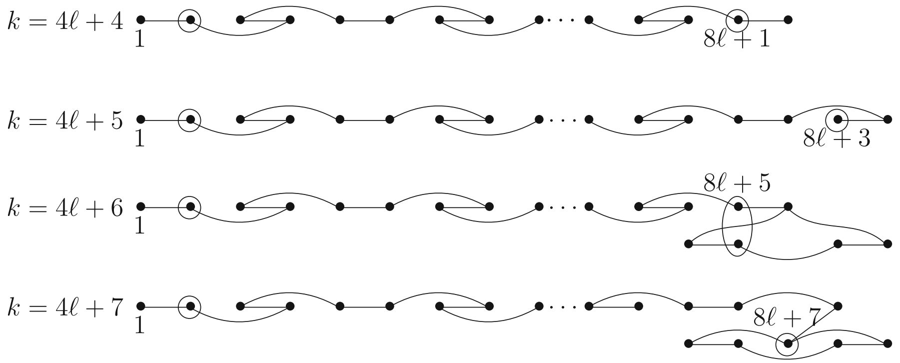

Fig. 1 Modules $\Sigma^{-2^k}M_k$

(2) $c_{18} := \mathrm{Sq}^1(x_4) \neq 0$, $\mathrm{Sq}^2(c_{18}) = 0$, $\mathrm{Sq}^2(x_4) = 0$, *and*
(3) $S(x_j) = (x_{j-1}^2, 0)$ *for* $j \geq 5$.

Proof We first introduce an intermediate set of generators w_j defined by

$$w_j = \begin{cases} u_j & j = 0, 1 \\ u_0u_1 + u_2 & j = 2 \\ u_1^{2^{j-2}}u_{j-2} + u_0^{2^{j-2}}u_{j-1} + u_j & j \geq 3. \end{cases}$$

These satisfy

$$S(w_j) = \begin{cases} (w_1, w_0^2) & j = 0 \\ (0, w_2 + w_0w_1) & j = 1 \\ (0, w_0w_2) & j = 2 \\ (w_{j-1}^2, 0) & j \geq 3. \end{cases}$$

Now we define $x_4 = w_0w_2^3 + w_4$ and, for $j \geq 5$

$$x_j = w_0^{2^{j-4}}w_2^{2^{j-3}}w_{j-2} + w_1^{3\cdot 2^{j-5}}w_2^{2^{j-4}}w_3^{2^{j-5}}w_{j-3} + w_0^{2^{j-5}}w_1^{2^{j-4}}w_2^{2^{j-3}}w_{j-3} + w_j.$$

One can check that these satisfy the claims of the lemma. □

Theorem 2.2 *For* $k \geq 4$ *there are* Q_0*-free* A_1*-submodules* $M_k \subset H^*(K_2)$ *with*

$$H_*(M_k; Q_1) = \begin{cases} \langle c_{18}, x_4 \rangle & k = 4 \\ \langle x_{k-1}^2, c_{18}x_4 \prod_{t=4}^{k-2} x_t^2 \rangle & k \geq 5. \end{cases}$$

The A_1*-module* $\Sigma^{-2^k}M_k$ *has the form in Fig. 1.*

Here and throughout $\langle s_1, \ldots, s_k \rangle$ denotes the span (resp. graded span) of elements in a vector space (resp. graded vector space). We depict A_1-modules with straight segments showing Sq^1, and curved segments Sq^2. We circle the Q_1-homology classes.

For example, if $k = 4\ell+4$, $\Sigma^{-2^k} M_k$ has a single nonzero class g_i for $1 \le i \le 8\ell+2$ with

$$\mathrm{Sq}^2\,\mathrm{Sq}^1\,\mathrm{Sq}^2(g_i) = \mathrm{Sq}^1\, g_{i+4} \ne 0 \text{ if } i \equiv 3\ (4),\ \ i \le 8\ell - 5,$$

and $\mathrm{Sq}^2\,\mathrm{Sq}^1(g_1) = \mathrm{Sq}^1(g_3) \ne 0$.

Proof of Theorem 2.2 We use the classes x_j, $j \ge 4$, of Lemma 2.1, but find it convenient to write c_{18} as x_3^2, even though it isn't a perfect square. In the discussion below we treat it as a perfect square. For $k \ge 4$, let $\mathbb{M}_k$ denote the finite A_1-submodule of H^*K_2 with basis all elements $\prod_{j=3}^{k} x_j^{e_j}$ satisfying $\sum e_j 2^j = 2^k$. Our desired A_1-module M_k will be a submodule of $\mathbb{M}_k$.

We first show that $\mathbb{M}_k$ is Q_0-free. Every monomial in $\mathbb{M}_k$ which is a perfect square can be written uniquely as $\prod_{s\in S} x_s^2 \cdot \prod_{t\in T} x_t^{2e_t}$ with $e_t > 1$ and S and T disjoint. It determines a Q_0-free summand

$$\prod_{t\in T} x_t^{2e_t-2} \cdot \bigotimes_{i\in S\cup T} \langle x_{i+1}, x_i^2\rangle.$$

Every monomial in $\mathbb{M}_k$ is in a unique one of these summands, as can be seen by writing the monomial as $P \cdot \prod_{u\in U} x_u$ with P a perfect square. This monomial is in the Q_0-free summand determined as above from $P \cdot \prod_{u\in U} x_{u-1}^2$.

We now show, somewhat similarly, that

$$H_*(\mathbb{M}_k;\, Q_1) = \langle x_{k-1}^2,\, x_4 \prod_{t=3}^{k-2} x_t^2\rangle.$$

Let $k \ge 5$, as the case $k = 4$ is elementary. Every monomial in $\mathbb{M}_k$ which is a perfect square or x_4 times a perfect square can be written uniquely as $\prod_{s\in S} x_s^2 \cdot x_4^{\varepsilon} \cdot \prod_{t\in T} x_t^{2e_t}$ with $e_t \ge 2$, S and T disjoint, and $\varepsilon \in \{0, 1\}$. Also, $T \ne \emptyset$ unless the monomial is x_{k-1}^2 or $x_3^2 x_4^3 x_5^2 \cdots x_{k-2}^2$, in order to have $\sum e_j 2^j = 2^k$. This monomial determines a Q_1-free summand

$$\prod_{s\in S} x_s^2 \cdot x_4^{\varepsilon} \cdot \prod_{t\in T} x_t^{2e_t-4} \bigotimes_{t\in T} \langle x_t^4, x_{t+2}\rangle.$$

Every monomial in $\mathbb{M}_k$ except x_{k-1}^2 and $x_3^2 x_4^3 x_5^2 \cdots x_{k-2}^2$ is in a unique one of these by writing it as

$$P \cdot x_4^{\varepsilon} \cdot \prod_{\substack{t\in T\\ t>2}} x_{t+2}$$

with P a perfect square; it is in the Q_0-free summand determined as above from $P \cdot x_4^{\varepsilon} \prod_{t\in T} x_t^4$.

By [13, Proposition 13.13 and p.203], the A_1-module $\mathbb{M}_k$ has an expression, unique up to isomorphism, as $M_k \oplus F$, with F free and M_k reduced. This M_k is Q_0-free and has the Q_1-homology stated in the theorem. To get a sense of why this is true, it is impossible for a Q_0-free module to have just one Q_1-homology class. Thus the two Q_1-homology classes must be in the same summand and what is left must be free over A_1.

We will determine its precise structure.

The module $\mathbb{M}_4$ has only the classes $\langle x_4, x_3^2\rangle$, so this is also M_4. For $k \geq 5$, $\mathbb{M}_k$ in gradings $\leq 2^k + 4$ has just the classes $\langle x_k, x_{k-1}^2, x_{k-2}^2 x_{k-1}, x_{k-2}^4\rangle$, in which Sq^1 and Sq^2 act as depicted on the left four dots in each row of Fig. 1. We will use Yu's Theorem ([4, Theorem 7.1]) to show that M_k must have the form claimed in the theorem. We thank Bob Bruner for suggesting the use of Yu's Theorem.

For $k \geq 5$, let $\mathbb{M}_k^*$ denote the A_1-module dual to $\mathbb{M}_k$. Its top class x_k^* is in grading $-2^k - 1$ and bottom class $(x_3^{2^{k-3}})^*$ is in grading $-2^k - 2^{k-3}$. Let $(\mathbb{M}_k^*)^+$ denote an A_1-module which agrees with $\mathbb{M}_k^*$ in gradings less than -2^k and for $i \geq -2^k$ has a single nonzero class y_i in grading i, with $\mathrm{Sq}^2\,\mathrm{Sq}^3\, y_{4j} = y_{4j+5} = \mathrm{Sq}^1\, y_{4j+4}$, and $0 \neq \mathrm{Sq}^1\, y_{-2^k} \in \mathrm{im}(\mathrm{Sq}^2)$. This $(\mathbb{M}_k^*)^+$ is Q_0-free and has a single nonzero Q_1-homology class, dual to $x_4 \prod_{t=3}^{k-2} x_t^2$, in grading $7 - 2^k - 2k$. By [13, Proposition 13.13 and p.203], $(\mathcal{M}_k^*)^+$ is isomorphic to the direct sum of a reduced module R and a free module. Since R is Q_0-free and reduced with a single nonzero Q_1-homology class, by Yu's Theorem, R is isomorphic to a shifted version of one of the four modules P_i, $0 \leq i \leq 3$, depicted in [4, Figure 1]. These modules begin with a form dual to one of the four endings of the modules in Fig. 1, followed by an infinite string of $\mathrm{Sq}^1\, z_n = \mathrm{Sq}^2\,\mathrm{Sq}^3\, z_{n-4}$.

Our module M_k is defined as the dual of R/T, where T is the submodule of R consisting of classes of grading $\geq -2^k$. This M_k will begin the same way as $\mathbb{M}_k$, as $\Sigma^{2^k}\langle g_1, \mathrm{Sq}^1\, g_1, g_3, \mathrm{Sq}^1\, g_3 = \mathrm{Sq}^2\,\mathrm{Sq}^1\, g_1\rangle$, and will end with one of the four types in Fig. 1, although *a priori* it could have a different length. Its top Q_1-homology class is in grading $2^k + 2k - 7$.

Since A_1 has 8 basis elements, the total number of basis elements in M_k will be congruent mod 8 to the number in $\mathbb{M}_k$. There is a 1–1 correspondence between a basis for $\mathbb{M}_k$ and the set of partitions of 2^{k-3} into 2-powers. (e_j tells the number of occurrences of 2^{j-3}.) It is proved in [6, Theorem 2] that this number of partitions is $\equiv 2 \bmod 8$ if k is even, and is $\equiv 4 \bmod 8$ if k is odd.

Let $k = 4\ell + 4$. The first module in Fig. 1 is the only possibility that satisfies that the top Q_1-homology class is in grading $2^k + 2k - 7 = 2^k + 8\ell + 1$ and the number of basis elements is $\equiv 2 \bmod 8$. The second and fourth types in Fig. 1 have their top Q_1-homology class in grading 3 mod 4, while if the third type had its top Q_1-homology class in $2^k + 8\ell + 1$, its number of basis elements would be 6 mod 8. A similar analysis, utilizing top Q_1-homology class mod 4 and number of basis elements mod 8, shows the M_k must be as claimed. □

Prior to discovering this proof, we had laboriously found explicit bases for M_k for $k \leq 9$. For example, with $abcd$ denoting $x_6^a x_5^b x_4^c c_{18}^d$, the basis for M_7 had x_7, 2000, 1200, 0400, 1040, 0240, 0080 along the top, as pictured in Fig. 1, and $1111 + 0320$, $0311 + 1031 + 1102 + 0240$, $0231 + 1022 + 0160$, $0151 + 1013 + 0222 + 0080$, and $0071 + 0142 + 0213 + 1004$ along the bottom.

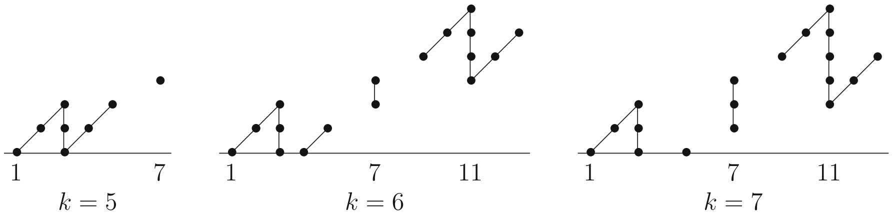

Fig. 2 $\mathrm{Ext}_{A_1}(\Sigma^{-2^k}M_k,\mathbb{Z}_2)$

3 Ext charts and tensor products

There is a nice pattern to the charts $\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{-2^k}M_k,\mathbb{Z}_2)$, depicted, as usual, in coordinates $(t-s,s)$. They are similar to familiar charts of $\mathrm{Ext}_{A_1}(H^*P^{2n},\mathbb{Z}_2)$ (e.g., [7]). In fact, there are A_1-module isomorphisms $\Sigma^{-2^{4\ell+4}}M_{4\ell+4}\approx H^*P^{8\ell+2}$ and $\Sigma^{-2^{4\ell+5}}M_{4\ell+5}\approx H^*P^{8\ell+4}$. For all k, all classes in these charts are v_1^4-periodic; i.e., $\mathrm{Ext}^{s,t}\to\mathrm{Ext}^{s+4,t+12}$ is bijective for $s\geq 0$. All the charts have the same upper edge, $(8i+x,4i+y)$ for $(x,y)=(1,0)$, $(2,1)$, $(3,2)$, and $(7,3)$. The lower edge drops by 1 for each increase in k, as long as $s\geq 0$. In Fig. 2 we show the beginning of the charts for $5\leq k\leq 7$. These Ext charts are easily obtained by standard methods from the explicit description of the modules in Fig. 1. See [5, Appendix A] for a rather detailed discussion of these methods.

Explicitly, $\Sigma^{-2^k}M_k$ has, for $i\geq 0$,

- 0 in $8i+6,8$,
- $\mathbb{Z}_2$ in $8i+1,2$ of filtration $4i+0,1$,
- $\mathbb{Z}_2$ in $8i+4,5$ of filtration $4i-k+6,7$ if $4i-k+6,7\geq 0$, else 0,
- $\mathbb{Z}/2^{k-4}$ in $8i+7$ with generator of filtration $4i-k+8$ if $4i-k+8\geq 0$, else $\mathbb{Z}/2^{4i+4}$ with generator of filtration 0, and
- $\mathbb{Z}/2^{k-2}$ in $8i+3$ with generator of filtration $4i-k+5$ if $4i-k+5\geq 0$, else $\mathbb{Z}/2^{4i+3}$ with generator of filtration 0.

Here, as usual, d dots connected by vertical segments yield a $\mathbb{Z}/2^d$ summand.

The A_1-modules M_k in Sect. 2 correspond to the E_1-modules M_k in the E_1-splitting of $H^*(K_2)$ in [9, (2.16)]. The correspondence is that, as an E_1-module, the A_1-module M_k is isomorphic to the E_1-module M_k plus perhaps a single copy of E_1. Moreover, the Q_1-homology classes agree, with u_j replaced by x_j. Also involved in the E_1 splitting in [9, (2.16)] were summands $M_k\cdot P$, where P is a product of finitely many distinct classes u_j^2 with $j\geq k$. Although u_j^2 is acted on trivially by E_1, $\mathrm{Sq}^2(u_j^2)\neq 0$, so the corresponding A_1 summands must do more than just multiply by the product of the classes u_j^2. To maintain some consistency with [9], in Definition 3.2 we will define M_kz_j to be a reduced Q_0-free A_1-module with

$$H_*(M_kz_j;Q_1)=H_*(M_k;Q_1)\otimes\langle u_{j+1}^2\rangle, \tag{3.1}$$

and similarly for products with more than one z_j.

For $j \geq 3$, let $G_j = \langle u_{j+2}, u_{j+1}^2, u_j^4 \rangle$ with $\mathrm{Sq}^2 \mathrm{Sq}^1 u_{j+2} = u_j^4$. If M is a Q_0-free A_1-module, then $M \otimes G_j$ is Q_0-free and

$$H_*(M \otimes G_j; Q_1) = H_*(M; Q_1) \otimes \langle u_{j+1}^2 \rangle.$$

Definition 3.2 We define $M_k z_j$ to be the reduced summand of the A_1-module $M_k \otimes G_j$.

Let P_j be the A_1-module for which there is a short exact sequence (SES)

$$0 \to G_j \to P_j \to \Sigma^{2^{j+2}-1} \mathbb{Z}_2 \to 0$$

with $u_{j+2} \in \mathrm{im}(\mathrm{Sq}^2)$. Then $H_*(P_j; Q_1) = 0$, so $M_k \otimes P_j$ is a free A_1-module by Wall's Theorem ([15, Theorem 2]), using also a Künneth Theorem for Q_i-homology. The short exact sequence of A_1-modules

$$0 \to M_k \otimes G_j \to M_k \otimes P_j \to \Sigma^{2^{j+2}-1} M_k \to 0 \tag{3.3}$$

has a long exact Ext sequence which implies that

$$\mathrm{Ext}_{A_1}^{s,t}(M_k \otimes G_j, \mathbb{Z}_2) \to \mathrm{Ext}_{A_1}^{s+1,t+1}(\Sigma^{2^{j+2}} M_k, \mathbb{Z}_2)$$

is bijective for $s \geq 1$ and surjective for $s = 0$. We deduce that, for the reduced submodule, $\mathrm{Ext}_{A_1}(M_k z_j, \mathbb{Z}_2)$ is formed from $\mathrm{Ext}_{A_1}(\Sigma^{2^{j+2}} M_k, \mathbb{Z}_2)$ by shifting filtrations down by 1, or, equivalently, by killing classes of filtration 0. Elements in the kernel of (3.3) when $s = 0$ correspond to free summands, which do not appear in the reduced submodule. Iterating, we have

Proposition 3.4 *For distinct* $j_i \geq k - 1$, $\mathrm{Ext}_{A_1}(M_k z_{j_1} \cdots z_{j_r}, \mathbb{Z}_2)$ *is formed from* $\mathrm{Ext}_{A_1}(\Sigma^{2^{j_1+2}+\cdots+2^{j_r+2}} M_k, \mathbb{Z}_2)$ *by reducing filtrations by* r.

The E_1-splitting of $H^* K_2$ in [9, Proposition 2.11] also involved products of modules with a class called u_2^2 there, but would be u_0^2 in our notation. Again, since $\mathrm{Sq}^2(u_0^2) \neq 0$, we must expand to an A_1-submodule of $H^*(K_2)$, namely

$$U = \langle u_0, u_1, u_0^2, u_2, u_1^2 \rangle. \tag{3.5}$$

The A_1-structure of this is $\Sigma^2 \langle 1, \mathrm{Sq}^1, \mathrm{Sq}^2, \mathrm{Sq}^2 \mathrm{Sq}^1, \mathrm{Sq}^3 \mathrm{Sq}^1 \rangle$, sometimes called the Joker ([3]). Note that $H_*(U; Q_1) = \langle u_0^2 \rangle$.

Proposition 3.6 *If* M *is a* Q_0*-free* A_1*-module and* U *is as above, then for* $s > 0$

$$\mathrm{Ext}_{A_1}^{s,t}(U \otimes M, \mathbb{Z}_2) \approx \mathrm{Ext}_{A_1}^{s+2,t+2}(M, \mathbb{Z}_2).$$

Proof There is a SES of A_1-modules

$$0 \to G \to F \to U \to 0,$$

where F is a free A_1-module on a generator of degree 2, and $G = \langle \iota_5, \mathrm{Sq}^2\,\iota_5, \mathrm{Sq}^3\,\iota_5\rangle$. After tensoring with M, the exact Ext sequence yields an isomorphism for $s > 0$

$$\mathrm{Ext}_{A_1}^{s,t}(G \otimes M, \mathbb{Z}_2) \to \mathrm{Ext}_{A_1}^{s+1,t}(U \otimes M, \mathbb{Z}_2).$$

Let $P = \Sigma^5 A_1/(\mathrm{Sq}^1)$. There is a SES of A_1-modules

$$0 \to \Sigma^{10}\mathbb{Z}_2 \to P \to G \to 0.$$

Then $P \otimes M$ is free by Wall's theorem, since $H_*(P; Q_1) = 0$ and $H_*(M; Q_0) = 0$. So tensoring this sequence with M yields isomorphisms for $s > 0$

$$\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{10}M, \mathbb{Z}_2) \to \mathrm{Ext}_{A_1}^{s+1,t}(G \otimes M, \mathbb{Z}_2).$$

Combining the two yields

$$\mathrm{Ext}_{A_1}^{s,t}(\Sigma^{10}M, \mathbb{Z}_2) \approx \mathrm{Ext}_{A_1}^{s+2,t}(U \otimes M, \mathbb{Z}_2).$$

The Q_0-free module $U \otimes M$ has v_1^4-periodicity in Ext

$$\mathrm{Ext}_{A_1}^{s,t}(U \otimes M, \mathbb{Z}_2) \approx \mathrm{Ext}_{A_1}^{s+4,t+12}(U \otimes M, \mathbb{Z}_2)$$

for $s > 0$ by [1, Theorem 5.1]. This is isomorphic to $\mathrm{Ext}_{A_1}^{s+2,t+12}(\Sigma^{10}M, \mathbb{Z}_2) \approx \mathrm{Ext}_{A_1}^{s+2,t+2}(M, \mathbb{Z}_2)$. □

We let UM_k and $UM_k z_{j_1} \cdots z_{j_r}$ denote reduced modules after tensoring with U. By Proposition 3.6, their Ext charts are obtained from those of M_k or $M_k z_{j_1} \cdots z_{j_r}$ by decreasing filtrations by 2.

The summand S in [9, Proposition 2.11] is the reduced summand of tensor products of the summands of the type that we have been considering here with an E_1-module N with Q_1-homology class x_9. We have an analogous construction in the A_1 context.

Using the classes w_j in the proof of Theorem 2.1, let N be the A_1-module

$$N = \langle w_2, w_0 w_2, w_1 w_2, w_3, w_2^2\rangle.$$

This satisfies $\mathrm{Sq}^2\,\mathrm{Sq}^3(w_2) = \mathrm{Sq}^1(w_3) = w_2^2$ with $|w_2| = 5$. It has the property that if M is a Q_0-free A_1-module, then

$$\mathrm{Ext}_{A_1}(N \otimes M, \mathbb{Z}_2) \approx \mathrm{Ext}_{A_1}(\Sigma^9 M, \mathbb{Z}_2) \tag{3.7}$$

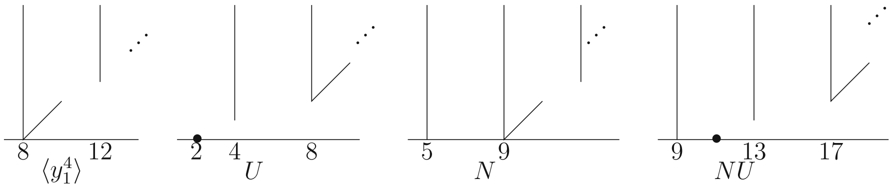

Fig. 3 $\mathrm{Ext}_{A_1}(-,\mathbb{Z}_2)$

in positive filtration as is easily seen from the Ext sequence obtained from the SES

$$0\to N'\otimes M\to N\otimes M\to\Sigma^9 M\to 0,$$

where N' is the A_1-submodule of N generated by w_2, since $N/N'=\Sigma^9\mathbb{Z}_2$ and $N'\otimes M$ is a free A_1-module by Wall's theorem. For any of our modules $U^\varepsilon M_k z_J$, we let $NU^\varepsilon M_k z_J$ denote a reduced submodule of $N\otimes U^\varepsilon M_k z_J$. It is isomorphic to $\Sigma^9 U^\varepsilon M_k z_J$.

The analogue of [9, Proposition 2.11] is given in Theorem 3.8. We let $y_1^2=u_0^4$; it is annihilated by Sq^1 and Sq^2.

Theorem 3.8 *There is an A_1-module splitting*

$$H^*K_2=P[y_1^2]\otimes(\mathbb{Z}_2\oplus U\oplus N\oplus NU)\otimes(\mathbb{Z}_2\oplus\bigoplus_{k\geq 4}M_k\text{Ł}_{k-1})\oplus F,$$

where F is free and $\text{Ł}_{k-1}=E[z_j:j\geq k-1]$ is an exterior algebra. The interpretation of $M_k z_{j_1}\cdots z_{j_r}$ is as in Definition 3.2, and $U\otimes M_k\text{Ł}_{k-1}$, $N\otimes M_k\text{Ł}_{k-1}$, and $NU\otimes M_k\text{Ł}_{k-1}$ mean the reduced summand. For reduced cohmology, one can remove the $\mathbb{Z}_2$ summand from the splitting.

Theorem 3.8 is obtained from [9, Proposition 2.11] by modifying the E_1 summands (where necessary) to make them A_1 modules that still retain the same Q_1 and Q_0 homologies.

Proof The correspondence with [9, Proposition 2.11] is $R\leftrightarrow\bigoplus M_k\text{Ł}_{k-1}$, $S=NR$, $\langle u_2^2\rangle\leftrightarrow U$, and $P[u_2^2]\leftrightarrow P[y_1^2]\otimes(\mathbb{Z}_2\oplus U)$. The Q_0- and Q_1-homology classes correspond and fill out the Q_i-homology of H^*K_2. The quotient of H^*K_2 by this large submodule is A_1-free by Wall's theorem. □

The E_2 page is obtained by applying $\mathrm{Ext}_{A_1}(-,\mathbb{Z}_2)$ to the summands of Theorem 3.8. Earlier in this section, we have done that for the summands involving $M_k\text{Ł}_{k-1}$. The others are small modules whose Ext is easily seen to be as in Fig. 3.
Here NU means the reduced summand of the A_1-module $N\otimes U$.

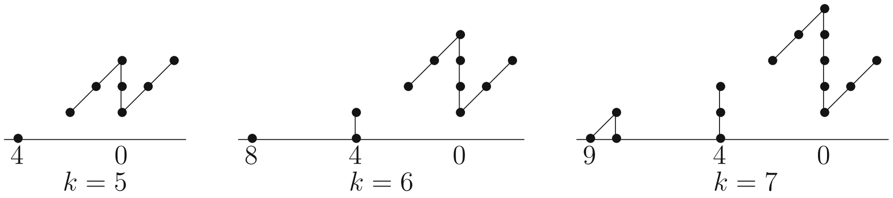

Fig. 4 $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, \Sigma^{-2^k} M_k)$

4 *ko*-cohomology and duality

Our main focus is on $ko_*(K_2)$, in part because of its relationship with Spin cobordism. In this short section, we explain briefly how we compute $ko^*(K_2)$ and the duality between it and $ko_*(K_2)$.

The Adams spectral sequence for $ko^{-*}(K_2)$ is obtained by applying $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, -)$ to the same A_1-modules used for $ko_*(K_2)$, with corresponding differentials. As we did for ku in [9], we display the ko-cohomology groups increasing from right to left.

In Fig. 4, we show the beginning of the charts for $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, M_k)$ for $k = 5, 6, 7$. This should be enough to suggest the entire pattern. These charts are the analogue of those in Fig. 2. They can be easily obtained from Fig. 1.

The analogue of Propositions 3.4 and 3.6 is as follows. It is proved using the exact sequences derived in Sect. 3.

Proposition 4.1 *(a). For distinct* $j_i \geq k$, $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, M_k z_{j_1} \cdots z_{j_r})$ *is formed from* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, \Sigma^{2^{j_1+2}+\cdots+2^{j_r+2}} M_k)$ *by increasing filtrations by r and extending to the left by* v_1^4*-periodicity.*

(b). If M is a Q_0*-free* A_1*-module, then* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, U \otimes M)$ *is formed from* $\mathrm{Ext}_{A_1}(\mathbb{Z}_2, M)$ *by increasing fitrations by 2 and extending to the left using* v_1^4*-periodicity.*

Analogously to [9, Theorem 1.16], we have the following remarkable duality result, where the group on the right hand side is the Pontryagin dual.

Theorem 4.2 *There is an isomorphism of* ko_**-modules* $ko_*(K_2) \approx (ko^{*+6} K_2)^\vee$.

This is deduced from [12, Corollary 9.3] similarly to the ku proof in [11]. The subtlety of the result is suggested by the observation that there is nothing like it for the E_2 pages. We illustrate it in [8], in which differentials and extensions are determined.

References

1. Adams, J.F.: A periodicity theorem in homological algebra. Proc. Camb. Philos. Soc. **62**, 365–377 (1966)
2. Anderson, D.W., Brown, E.H., Jr., Peterson, F.P.: The structure of the Spin cobordism ring. Ann. Math. **86**, 271–298 (1967)
3. Baker, A.: Iterated doubles of the Joker and their realisability. Homol. Homotopy Appl. **20**, 341–360 (2018)
4. Bruner, R.R.: Idempotents, localizations, and Picard groups of $A(1)$-modules. Contemp. Math. AMS **617**, 81–108 (2014)
5. Bruner, R.R., Greenlees, J.P.C.: Connective real K-theory of finite groups. Math Surv. Monogr. Am. Math. Soc. **169** (2010)
6. Churchhouse, R.F.: Congruence properties of the binary partition function. Math. Proc. Camb. Philos. Soc. **66**, 365–370 (1968)
7. Davis, D.M.: Generalized homology and the generalized vector field problem. Q. J. Math. Oxf. **25**, 169–193 (1974)
8. Davis, D.M.: The connective KO theory of the Eilenberg MacLane space $K(\mathbb{Z}_2, 2)$. arXiv:2502.14982
9. Davis, D.M., Wilson, W.S.: The connective K-theory of the Eilenberg–MacLane space $K(\mathbb{Z}_2, 2)$. Glasgow Math. J. **66**, 188–220 (2024)
10. Davis, D.M., Wilson, W.S.: Stiefel-Whitney classes and nonimmersions of orientable and Spin manifolds. Topol. Appl. **307**, 107780 (2022)
11. Davis, D.M., Greenlees, J.: Gorenstein duality and universal coefficient theorems. J. Pure Appl. Algorithm. **227**, 107265 (2023)
12. Mahowald, M., Rezk, C.: Brown-Comenetz duality and the Adams spectral sequence. Am. J. Math. **121**, 1153–1177 (1999)
13. Margolis, H.R.: Spectra and the Steenrod algebra, North-Holland Math Library **29** (1983)
14. Serre, J.P.: Cohomologie modulo 2 des complexes d'Eilenberg MacLane. Comment. Math. Helv. **27**, 198–231 (1953)
15. Wall, C.: A characterization of simple modules over the Steenrod algebra mod 2. Topology **1**, 249–254 (1962)
16. Wilson, W.S.: A new relation on the Stiefel–Whitney classes of Spin manifolds. Ill. J. Math. **17**, 115–127 (1973)