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# THE CONNECTIVE MORAVA $K$ -THEORY OF THE SECOND MOD $p$ EILENBERG-MACLANE SPACE

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ABSTRACT. We develop tools for computing the connective  $n$ -th Morava  $K$ -theory of spaces. Starting with a Universal Coefficient Theorem that computes the cohomology version from the homology version, we show that every step in the process of computing one is mirrored in the other and that this can be used to make computations. As our example, we compute the connective  $n$ -th Morava  $K$ -theory of the second mod  $p$  Eilenberg-MacLane space.

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## 1. INTRODUCTION

Being able to compute is central to much of algebraic topology. Computing generalized (co)homology theories of basic spaces usually runs from difficult to impossible. One exception has been the extraordinary  $K$ -theories of Jack Morava,  $K(n)_*(X)$ . They have a Künneth isomorphism that makes them more tractable to computations than most.

There is a connective version of Morava  $K$ -theories,  $k(n)_*(X)$ , and in this paper we make some progress towards computing with this. In particular, we develop some tools that can be applied to this problem in general, and then we apply them to compute the  $n$ th connective Morava  $K$ -theory of the second mod  $p$  Eilenberg-MacLane space,  $K_2 = K(\mathbf{Z}/p, 2)$ , where  $\mathbf{Z}/p$  is the integers modulo the prime  $p$ .

Anderson-Hodgkin [AH68] showed that  $K(1)_*(K_2)$  was trivial. The third author searched, periodically over the decades, for the differentials in the Atiyah-Hirzebruch spectral sequence that would reduce the already small group  $H_*(K_2; K(1)_*)$  to zero at  $p = 2$ . The differentials in the Atiyah-Hirzebruch spectral sequence are the same as those in the Adams spectral sequence, so this paper finally gives the third author great satisfaction. The project grew into this paper.

The main computation of the paper is to compute both  $k(n)_*(K_2)$  and  $k(n)^*(K_2)$  as  $k(n)_*$  (and  $k(n)^*$ ) modules. The  $n = 1$  case is essential for the first and third authors' determination of  $ku^*(K_2)$  and  $ku_*(K_2)$  for all primes in [DW24b]. It was also useful in the first author's determination of  $ko^*(K_2)$  and  $ko_*(K_2)$  in [D], with the  $E_2$  page of the Adams spectral sequence for this computed in [DW24a].

One of our main tools is obtained by combining results of Robinson and Lazarev for computational purposes.

**Theorem 1.1.** *For  $X$  of finite type with  $K(n)_*(X)$  finitely generated over  $K(n)_*$ , there is a universal coefficient spectral sequence that collapses:*

$$\mathrm{Ext}_{k(n)_*}^{s,t}(k(n)_*(X), k(n)_*) \Rightarrow k(n)^{s+t}(X)$$

In [Rob87] Alan Robinson created the universal coefficient spectral sequence for homology theories satisfying certain hypotheses. These were shown to be satisfied by  $k(n)_*$  by him in [Rob89] and later by Andrey Lazarev in [Laz01]. We will show the universal coefficient spectral sequence collapses in this case.

From this result, we derive the next important tool.

**Theorem 1.2 (The Pairing).** *For  $X$  of finite type with  $K(n)_*(X)$  finitely generated over  $K(n)_*$ , there is a differential  $d^r(\alpha) = v^r\beta$  in the Adams spectral sequence for  $k(n)^*(X)$  if and only if there is a corresponding  $d_r(\beta') = v^r\alpha'$  in the Adams spectral sequence for  $k(n)_*(X)$ , with  $|\alpha| = |\alpha'|$  and  $|\beta| = |\beta'|$ .*

It is the interaction between  $k(n)$  cohomology and homology from these two results that allows us to do our computation. Theorem 1.1 gives a duality of sorts between  $k(n)_*(X)$  and  $k(n)^*(X)$ , but Theorem 1.2 goes even further and says that there is a duality every step of the way in the computation. In our case, we have that  $K_2$  is an H-space so both Adams spectral sequences are multiplicative. Although this does not give us a Hopf algebra, there is enough similarity in the structure that we can make good use of it.

The plan of the paper is to state the results of the main computation in the next section. We set up some notation in Section 3. In Section 4 we compute the  $E_2$  term of the Adams spectral sequence for  $k(n)^*(K_2)$ . In Section 5 we illustrate its behavior for  $n = 2$ . We give some necessary definitions and numbers in Section 6. In Section 7, we prove the two theorems in the introduction and establish some other preliminaries we need. All the hard work is done in Section 8 where the differentials are computed. The results for  $k(n)_*(K_2)$  are all collected in Section 9 and the final section is devoted to describing the results at  $p = 2$ .

## 2. STATEMENT OF RESULTS

In this section we define only what we need to efficiently state the results of our main computation of  $k(n)^*(K_2)$ . Many details will be properly developed later.

**2.1. Basic notation.** All our cohomology and homology groups will be mod  $p$ . The connective  $n$ th Morava  $K$ -theory spectrum,  $k(n)$ , has  $k(n)^* = \mathbf{Z}/p[v_n]$  with  $|v_n| = -2(p^n - 1)$ .

We let  $P(x)$ ,  $E(x)$ , and  $\Gamma(x)$  be the polynomial, exterior, and divided power algebras on  $x$  (which could be a single generator or a set of generators) over  $\mathbf{Z}/p$ . In addition, we need the truncated polynomial algebra,  $T_k(x) := P(x)/(x^k)$ , and its dual,  $\Gamma_k(x)$ .

The divided power algebra  $\Gamma(x)$  for a single  $x$  is additively generated by elements  $\gamma_i(x)$  for  $i \geq 0$  (the divided powers of  $x$ ) with  $|\gamma_i(x)| = i|x|$  and

$$\gamma_i(x)\gamma_j(x) = \binom{i+j}{i} \gamma_{i+j}(x).$$

As an algebra,  $\Gamma(x)$  is  $T_p(\gamma_{p^m}(x) : m \geq 0)$ . For  $p = 2$ , this is an exterior algebra.

For a rational number  $x$ ,  $\lfloor x \rfloor$ , the *floor* of  $x$ , denotes the largest integer not exceeding  $x$ , and  $\lceil x \rceil$ , the *ceiling* of  $x$ , denotes the smallest integer not exceeded by  $x$ .

For a graded connected  $\mathbf{Z}/p$ -algebra  $A$ , we let  $\bar{A}$  denote the augmentation ideal of  $A$ , its vector space of positive degree elements.

**2.2. The mod  $p$  cohomology of  $K_2 := K(\mathbf{Z}/p, 2)$ .** In what follows, *all tensor products are over  $\mathbf{Z}/p$  unless otherwise stated.*

To compute with the Adams spectral sequence, we need (for  $p$  an odd prime)

$$(2.1) \quad \begin{aligned} H^* K_2 &= P(\iota_2) \otimes P(z_i : i > 0) \otimes E(u_i : i \geq 0) \\ \text{with } |\iota_2| &= 2, \quad |z_i| = 2p^i + 2 \quad \text{and} \quad |u_i| = 2p^i + 1. \end{aligned}$$

Let  $y_{n,j} = \iota_2^{p^j}$ . In particular,  $\iota_2 = y_{n,0}$  and  $\iota_2^p = y_{n,0}^p = y_{n,1}$ . In general,  $y_{n,j}^p = y_{n,j+1}$  with  $|y_{n,j}| = 2p^j$ .

For  $p = 2$ ,  $H^* K_2$  has a similar description but with  $u_i^2 = z_{i+1}$ . We will say more about this in Section 10.

We define  $w_{n,i} \in H^* K_2$  for  $i \geq 0$  by

$$(2.2) \quad w_{n,i} := \begin{cases} u_n & \text{for } i = 0 \\ u_{n+i} - u_{n-i} z_i^{p^n - p^{n-i}} & \text{for } 0 < i \leq n \\ y_{n,i-n-1/2} z_i^{p^{n-1}} \\ = w_{n,i-n-1} y_{n,i-n-1}^{p-1} z_i^{p^{n-1}} & \text{for } i \geq n+1, \end{cases}$$

where

$$(2.3) \quad y_{n,i+1/2} := y_{n,i}^{p-1} w_{n,i} \quad \text{for integers } i \geq 0.$$

In general, all our variables, such as  $n, i, j, k, s$ , are non-negative integers. The number  $1/2$  arises often, and should be clear from context.

In Section 3 we will see that there is an Adams spectral sequence converging to  $k(n)^* K_2$  for which the input is  $k(n)^* \otimes H^* K_2$ . It is indexed in such a way that

- the filtration and dimension of  $v_n$  are 1 and  $-2(p^n - 1)$ ,
- the elements of  $H^* K_2$  have their usual positive degrees and Adams filtration 0, and
- differentials  $d^r$  raise rather than lower degree by 1, while raising filtration by  $r$ .

In Section 4 we will see that the action of the Milnor primitive  $Q_n$  on  $H^* K_2$  gives us  $d^1(x) = v_n Q_n(x)$ . From this we get

$$(2.4) \quad \begin{aligned} d^1(y_{n,0}) &= v_n u_n \\ \text{and } d^1(u_s) &= \begin{cases} v_n z_{n-s}^{p^s} & \text{for } 0 \leq s < n \\ 0 & \text{for } s = n \\ v_n z_{s-n}^{p^n} & \text{for } s > n. \end{cases} \end{aligned}$$

This implies that for  $w_{n,s}$  as in (2.2),  $d^1(w_{n,s}) = 0$  for  $0 \leq s \leq n$ . We can regard  $w_{n,s}$  for such  $s$  as a substitute for  $u_{n+s}$  that survives to  $E_2$ .

In Section 8 for  $p$  odd and Section 10 for  $p = 2$ , we will see that there are higher Adams differentials

$$(2.5) \quad \begin{aligned} d^{\rho_n(i)}(y_{n,i}) &= v_n^{\rho_n(i)} w_{n,i} \\ \text{and} \quad d^{\rho_n(i+1/2)}(y_{n,i+1/2}) &= v_n^{\rho_n(i+1/2)} z_{n+i+1} \end{aligned}$$

for integers  $i \geq 0$ , where the numbers  $\rho_n(i)$  and  $\rho_n(i+1/2)$  are given in Lemma 6.4. The latter are uniquely determined by the dimensions of the elements in question. For integers  $0 \leq i \leq n$ , they are

$$\rho_n(i) = p^i \quad \text{and} \quad \rho_n(i+1/2) = (p-1)p^i.$$

In particular  $\rho_n(0) = 1$ , so the first differential of (2.5) for  $i = 0$  coincides with the first differential of (2.4).

**2.3. The effect of the Adams  $d^1$ .** *The additional  $d^1$ s of (2.4) make the passage from  $E_1$  to  $E_2$  more complicated than the passage to higher terms brought about by the higher differentials of (2.5). We will outline these processes here in order to motivate the complicated expressions in Theorem 2.12, our main computational result.*

In order to work out the implications of (2.4), the following additive isomorphisms and definitions for each positive  $n$  and  $i \geq 0$  are convenient.

$$(2.6) \quad \begin{aligned} P(y_{n,i}) &\cong T_p(y_{n,i}) \otimes P(y_{n,i+1}), \\ E(u_s : s \geq 0) &\cong EE_n \otimes E(w_{n,0}) \otimes W_{n,0}, \text{ where} \\ EE_n &:= E(u_s : 0 \leq s < n) \otimes E(u_{2n+s} : s > 0) \quad \text{and} \\ W_{n,i} &:= E(w_{n,i+s} : 1 \leq s \leq n) \quad \text{for } w_{n,i+s} \text{ as in (2.2),} \\ P(z_s : s > 0) &\cong L_n \otimes TZ_{n,0} \otimes PZ_n, \text{ where} \\ L_n &:= \bigotimes_{0 < s < n} T_{p^{n-s}}(z_s), \\ TZ_{n,i} &:= T_{p^n}(z_{n+s} : s > i), \quad \text{and} \\ PZ_n &:= P(z_s^{e_n(s)} : s > 0) \quad \text{with } e_n(s) := \begin{cases} p^{n-s} & \text{for } 0 < s \leq n \\ p^n & \text{for } s > n. \end{cases} \end{aligned}$$

We will make use of these  $W_{n,i}$  and  $TZ_{n,i}$  for  $i > 0$  later.

**Remark 2.7.** Although stated as additive isomorphisms, much of the algebra structure is preserved and we need it. For example, the additive isomorphism for  $P(y_{n,i})$  comes from the multiplicative extension

$$P(y_{n,i+1}) \longrightarrow P(y_{n,i}) \longrightarrow T_p(y_{n,i}),$$

meaning that  $P(y_{n,i})$  is a free module over the subring  $P(y_{n,i+1})$  and

$$T_p(y_{n,i}) = P(y_{n,i}) \otimes_{P(y_{n,i+1})} \mathbf{Z}/p.$$

Here, if we have  $d^r(y_{n,i}) \neq 0$ , we have  $d^r(y_{n,i}^p = y_{n,i+1}) = 0$ , but this requires the multiplicative structure. Similarly, we have

$$PZ_n \longrightarrow P(z_s : s > 0) \longrightarrow L_n \otimes TZ_{n,0}.$$

Putting this all together, we have

$$(2.8) \quad H^*K_2 \cong T_p(y_{n,0}) \otimes P(y_{n,1}) \otimes EE_n \otimes E(w_{n,0}) \otimes W_{n,0} \otimes L_n \otimes TZ_{n,0} \otimes PZ_n.$$

We see in (4.2) that the  $d^1$  of (2.4) on  $k(n)^*$  tensored with (2.8) is confined to  $k(n)^* \otimes D_1$  with

$$(2.9) \quad D_1 := T_p(y_{n,0}) \otimes E(w_{n,0}) \otimes EE_n \otimes PZ_n.$$

The  $d^1$  homology of  $k(n)^* \otimes T_p(y_{n,0}) \otimes E(w_{n,0})$  is just  $k(n)^* \otimes E(y_{n,1/2}) \oplus T_{p-1}(y_{n,0}) \otimes \overline{E(w_{n,0})}$ . This is illustrated in the top diagram of (8.3) for  $j = 0$ , where  $\rho_n(0) = 1$ . The  $d^1$  homology of  $k(n)^* \otimes EE_n \otimes PZ_n$  is  $k(n)^* \oplus L_n \otimes TZ_{n,0}$ . Combining these results is tricky so we just observe that the elementary  $v_n$ -torsion in (2.9) is the image of  $Q_n$ , i.e.  $Q_n(D_1)$ . If we remove the elements in  $E(y_{0,1/2})$  from (2.9), what remains is free over  $E(Q_n)$ .

It is straightforward to compute the Poincaré series from this information. Although this does not give an explicit base, it isn't hard to filter it and get a basis for the associated graded version. For example, for  $EE_n \otimes PZ_n$ , we would have

$$(2.10) \quad \bigoplus_{0 < k \leq n} E(u_i : 0 \leq i < n - k) \otimes E(u_i : 2n < i) \otimes \overline{P(z_k^{e_n(k)})} \otimes P(z_i^{e_n(i)} : i > k) \\ \oplus \bigoplus_{2n < k} E(u_i : i > k) \otimes \overline{P(z_{k-n}^{p^n})} \otimes P(z_i^{p^n} : i > k - n)$$

We define

$$(2.11) \quad \begin{aligned} S_{n,0} &:= Q_n(D_1) \\ M_{n,i} &:= P(y_{n,i+1}) \otimes W_{n,i} \otimes L_n \otimes TZ_{n,i} \quad \text{for } i \geq 0. \\ M_{n,i+1/2} &:= P(y_{n,i+1}) \otimes W_{n,i} \otimes L_n \otimes TZ_{n,i+1} \quad \text{for } i \geq 0, \\ S_{n,i} &:= T_{\rho_n(i)}(v_n) \otimes T_{p-1}(y_{n,i}) \otimes \overline{E(w_{n,i})} \\ &= T_{\rho_n(i)}(v_n) \otimes \{w_{n,i}y_{n,i}^s : 0 \leq s \leq p-2\} \quad \text{for } i > 0, \\ \text{and} \quad S_{n,i+1/2} &:= T_{\rho_n(i+1/2)}(v_n) \otimes \overline{T_{p^n}(z_{n+i+1})} \\ &= T_{\rho_n(i+1/2)}(v_n) \otimes \{z_{n+i+1}^s : 1 \leq s < p^n\} \quad \text{for } i \geq 0. \end{aligned}$$

These will figure in Theorem 2.12. Using this notation, we have computed the elementary  $v_n$ -torsion as  $s_{n,0} \otimes M_{n,0}$ .

**2.4. The effect of higher Adams differentials.** The higher differentials of (2.5) are easier to deal with since each is nonzero on a single multiplicative generator. They are illustrated in the diagrams of (8.3) below.

The vector spaces  $S_{n,i}$  and  $S_{n,i+1/2}$  of (2.11) can also be written as

$$S_{n,i} = d^{\rho_n(i)} \left( \overline{T_p(y_{n,i})} \right) / v_n^{\rho_n(i)}$$

and

$$S_{n,i+1/2} = d^{\rho_n(i+1/2)} \left( \overline{E(y_{n,i+1/2})} \otimes T_{p^n-1}(z_{n+i+1}) \right) / v_n^{\rho_n(i+1/2)}.$$

We can now state the main computational result of this paper.

**Theorem 2.12.** *For an odd prime  $p$ ,  $k(n)^*(K_2)$  has the following three summands as a  $k(n)^*$ -module:*

- (i) *The  $k(n)^*$  free summand,  $k(n)^* \otimes L_n$ , for  $L_n$  as in (2.6).*
- (ii) *The higher torsion summand,*

$$\bigoplus_{\ell > 0} (M_{n,\ell/2} \otimes S_{n,\ell/2}),$$

*for  $M_{n,\ell/2}$  and  $S_{n,\ell/2}$  as in (2.11).*

- (iii) *The elementary torsion summand,  $S_{n,0} \otimes M_{n,0}$  as in (2.11).*

**Remark 2.13. The  $v_n$ -torsion free summand.** Inverting  $v_n$  kills all but the first summand of  $k(n)^*K_2$ , which becomes  $K(n)^*(K_2)$ , as described in [RW80, dual to Theorem 11.1]. This  $k(n)^*$ -free summand is all that appears in negative degrees, where it is finite in each degree. In addition, every positive degree is also finite.

**Remark 2.14. The multiplicative structure.** Theorem 2.12 describes a ring as well as a  $k(n)^*$ -module, but we can only show that the ring structure is that of the Adams  $E_\infty$ -term. We cannot rule out nontrivial multiplicative extensions. For  $n > 2$ , we cannot show by dimensional arguments that  $v_n z_{n+i}^{p^n} = 0$  for  $i > 0$ . Let

$$\kappa_n = \prod_{0 < i < n} z_{n-i}^{p^i-1},$$

the top class in  $L_n$ . We can show by induction on  $n$  that  $|\kappa_n| = 2(p^n - 1)(n - 1)$  using the identity

$$\kappa_{n+1} = \kappa_n^p (z_1 z_2 \cdots z_n)^{p-1}.$$

We cannot rule out the multiplicative extension

$$v_n z_{n+i}^{p^n} = v_n^{n-1} \kappa_n z_{2n+i}$$

(note that  $|v_n z_{n+i}^{p^n}| = |z_{2n+i}|$  and  $|v_n^{n-1} \kappa_n| = 0$  for  $n > 2$  and  $i > 0$ ).

## 3. OUR ADAMS SPECTRAL SEQUENCE NOTATION

The  $k(n)$  under consideration here is the the connective version of Morava's  $n$ th extraordinary  $K$ -theory  $K(n)$ . We have

$$\begin{aligned} \pi_* k(n) &= P(v_n) && \text{with } |v_n| = 2(p^n - 1) \\ \text{and } H^*(k(n)) &= \mathcal{A}/\mathcal{A}(Q_n), \end{aligned}$$

where  $\mathcal{A}$  is the mod  $p$  Steenrod algebra and  $Q_n$  is the  $n$ th Milnor primitive.

We have

$$(3.1) \quad k(n)^i X := [X, k(n)]_{-i} = \pi_{-i} F[X, k(n)] \quad \text{and} \quad k(n)_i X := \pi_i(k(n) \wedge X).$$

Given  $\alpha \in k(n)_i X$  represented by a map  $S^i \rightarrow k(n) \wedge X$ , and  $\beta \in k(n)_j X$  represented by a map  $X \rightarrow \Sigma^j k(n)$ , we get an element  $\langle \alpha, \beta \rangle \in \pi_{i-j} k(n)$ , which is the composite

$$S^i \xrightarrow{\alpha} k(n) \wedge X \xrightarrow{k(n) \wedge \beta} k(n) \wedge \Sigma^j k(n) \xrightarrow{m} \Sigma^j k(n),$$

where  $m : k(n) \wedge k(n) \rightarrow k(n)$  is the multiplication in the ring spectrum  $k(n)$ .

The groups of (3.1) can be computed with the Adams spectral sequence as follows. In the above,  $k(n)$  is the spectrum representing connective Morava  $K$ -theory and  $F(X, Y)$  denotes the function spectrum for maps of spectra  $X \rightarrow Y$ . The ring structure on  $k(n)$  allows us to extend the map  $S^{|v_n|} \rightarrow k(n)$  representing  $v_n$  to the map  $v_n$  in the fiber sequence

$$(3.2) \quad \Sigma^{|v_n|} k(n) \xrightarrow{v_n} k(n) \xrightarrow{j} H/p \xrightarrow{\delta} \Sigma^{2p^n-1} k(n)$$

where  $H/p$  is the mod  $p$  Eilenberg-Mac Lane spectrum. The composite

$$(3.3) \quad H/p \xrightarrow{\delta} \Sigma^{2p^n-1} k(n) \xrightarrow{j} \Sigma^{2p^n-1} H/p$$

is the Milnor primitive operation operation  $Q_n$ .

With the maps in (3.2) we can construct the following *Adams diagram* for  $k(n)$ ,

$$(3.4) \quad \begin{array}{ccccccc} k(n) & \xleftarrow{v_n} & \Sigma^{|v_n|} k(n) & \xleftarrow{v_n} & \Sigma^{2|v_n|} k(n) & \xleftarrow{v_n} & \Sigma^{3|v_n|} k(n) & \xleftarrow{\quad} & \cdots \\ j \downarrow & & j \downarrow & & j \downarrow & & j \downarrow & & \\ H/p & & \Sigma^{|v_n|} H/p & & \Sigma^{2|v_n|} H/p & & \Sigma^{3|v_n|} H/p & & \end{array}$$

Each fiber sequence

$$\Sigma^{(s+1)|v_n|} k(n) \rightarrow \Sigma^{s|v_n|} k(n) \rightarrow \Sigma^{s|v_n|} H/p$$



leads to a long exact sequence of homotopy groups. The same is true if we apply either the functor  $F(X, -)$ , the *cohomological case*, or  $(X \wedge -)$ , the *homological case*, to (3.4).

In each case these long exact sequences assemble into an exact couple (see [Rav86, §2.1]) leading to a spectral sequence  $\{E_r^{s,t}\}$ , where

- $E_1^{s,t}$  is either  $\pi_{t-s}(F(X, \Sigma^{s|v_n|} H/p)) = H^{s(2p^n-1)-t} X$ , the indicated mod  $p$  cohomology group of  $X$ , or  $\pi_{t-s}(X \wedge \Sigma^{s|v_n|} H/p) = H_{t-s(2p^n-1)} X$ , the indicated mod  $p$  homology group of  $X$ .
- $E_2^{s,t}$  is either  $\text{Ext}_{E(Q_n)}^{s,t}(\mathbf{Z}/p, H^* X)$  or  $\text{Ext}_{E(Q_n)}^{s,t}(H^* X, \mathbf{Z}/p)$ . This can be derived from (3.3).
- $d_r : E_r^{s,t} \rightarrow E_r^{s+r, t+r-1}$ . The filtration index  $s$  is raised by  $r$  and the dimension index  $t - s$  is lowered by one.
- $E_\infty^{s,t}$  is a certain subquotient of either  $k(n)^{s-t} X$  or  $k(n)_{t-s} X$ .

These are the *classical Adams spectral sequences* for  $k(n)^* X$  and  $k(n)_* X$ . It is common to depict them in a chart in which  $E_r^{s,t}$  has Cartesian coordinate  $(t - s, s)$ . Thus  $d_r$  is an arrow lowering the first coordinate by 1 and raising the second by  $r$ , making it a line with slope  $-r$ .

The *Adams spectral sequence* for  $k(n)^*(K_2)$  has

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^*(k(n)), H^* K_2) \cong \text{Ext}_{E(Q_n)}^{s,t}(\mathbf{Z}/p, H^* K_2) \implies k(n)^{-(t-s)}(K_2).$$

We use the usual grading for the Adams spectral sequence so that  $E_r^{s,t}$  is displayed with Cartesian coordinates  $(t - s, s)$ , but then we give the negative  $x$ -axis positive degrees, rewriting  $E_r^{s,t}$  as  $G_{s-t,s}^r$  in position  $(s-t, s)$ . We use  $d^r$  for our cohomology differentials. *In this depiction, differentials raise rather than lower the first coordinate by 1.*

We also have the *Adams spectral sequence* for  $k(n)_*(K_2)$ , and need to have distinct notation to clearly separate it from the cohomology notation. It has

$$E_2^{s,t} = \text{Ext}_{\mathcal{A}}^{s,t}(H^*(k(n) \wedge K_2), \mathbf{Z}/p) \cong \text{Ext}_{E(Q_n)}^{s,t}(H^* K_2, \mathbf{Z}/p) \implies k(n)_{t-s}(K_2).$$

We use the usual grading for the Adams spectral sequence so that  $E_r^{s,t}$  is at the Cartesian  $(t - s, s)$ . Here we don't need the negative grading, but to distinguish this from the cohomology Adams spectral sequence, we write  $E_r^{s,t}$  as  $G_r^{t-s,s}$  in position  $(t - s, s)$ . Here we use  $d_r$  for the differential so we can keep track of which is which.

To summarize, in the cohomological case

$$(3.5) \quad G_{x,y}^r := E_r^{y,y-x} \quad \text{with differentials } d^r : G_{x,y}^r \rightarrow G_{x+1,y+r}^r,$$

and in the homological case,

$$(3.6) \quad G_r^{x,y} := E_r^{y,y+x} \quad \text{with differentials } d_r : G_r^{x,y} \rightarrow G_r^{x-1,y+r}.$$

We need the  $E(Q_n)$ -module structure of  $H^*X$  (which we will describe in the next section) in order to compute the  $E_2$ -terms. Any  $E(Q_n)$ -module  $M$  is the sum of a free module and  $\mathbf{Z}/p$ -vector space on which  $Q_n$  acts trivially. As a result, it is easy to compute the relevant Ext groups. We have

$$(3.7) \quad \left\{ \begin{array}{l} \text{Ext}_{E(Q_n)}^{**}(\mathbf{Z}/p, \mathbf{Z}/p) = P(v_n) \quad \text{with } v_n \in \text{Ext}^{1,2p^n-1}, \\ \text{Ext}_{E(Q_n)}^{s,t}(E(Q_n), \mathbf{Z}/p) = \begin{cases} \mathbf{Z}/p & \text{for } (s,t) = (0,0) \\ 0 & \text{otherwise,} \end{cases} \\ \text{and } \text{Ext}_{E(Q_n)}^{s,t}(\mathbf{Z}/p, E(Q_n)) = \begin{cases} \mathbf{Z}/p & \text{for } (s,t) = (1, 2p^n - 1) \\ 0 & \text{otherwise.} \end{cases} \end{array} \right.$$

#### 4. THE $Q_n$ HOMOLOGY OF $H^*K_2$ AND THE ADAMS $E_2$ TERM

Following Tamanoi, [Tam99, Theorem 5.2], we have, at odd primes,  $u_i = Q_i t_2$  and  $z_i = Q_i u_0$  (in particular  $z_0 = 0$ ), giving us  $H^*K_2$  as in (2.1), where  $Q_i$  again is the  $i$ th Milnor primitive. Continuing to follow Tamanoi, we have

$$\begin{aligned} Q_n t_2 &= u_n \\ Q_n u_s &= \begin{cases} z_{n-s}^{p^s} & \text{for } 0 \leq s < n \\ 0 & \text{for } s = n \\ z_{s-n}^{p^n} & \text{for } s > n \end{cases} \\ Q_n z_s &= 0. \end{aligned}$$

To compute the  $Q_n$  homology we filter  $H^*K_2$  by powers of the ideal

$$(y_{n,0}, u_s, z_s^{e_n(s)} : s \geq 0),$$

where  $z_0 = 0$  and  $e_n(s)$  is as in (2.6). This means  $z_s \in F^0$  and  $z_s^{e_n(s)} \in F^1$  for  $s > 0$ , and  $u_s \in F^1$  for  $s \geq 0$ . The associated bigraded object  $E_0 H^*K_2$  and its  $Q_n$  homology are indicated in the following diagram, which uses the notation of

(2.6).

$$\begin{array}{ccc}
L_n \otimes TZ_{n,0} & \rightsquigarrow & L_n \otimes TZ_{n,0} \\
\otimes & & \otimes \\
P([y_{n,0}]) \otimes E([u_n]) & \rightsquigarrow & P([y_{n,1}]) \otimes E([y_{n,1/2}]) \\
\otimes & & \otimes \\
(4.1) \quad E([u_s] : 0 \leq s < n) \otimes P\left(\left[z_s^{p^{n-s}}\right] : 0 < s \leq n\right) & \rightsquigarrow & \mathbf{Z}/p \\
\otimes & & \otimes \\
E([u_{2n+s}] : s > 0) \otimes P\left(\left[z_{n+s}^{p^n}\right] : s > 0\right) & \rightsquigarrow & \mathbf{Z}/p \\
\otimes & & \otimes \\
E_0W_{n,0} & \rightsquigarrow & E_0W_{n,0},
\end{array}$$

where elements enclosed in square brackets are in  $E_0H^*K_2$  (where they are indecomposable) corresponding to unbracketed elements in  $H^*K_2$ , which may be decomposable. The elements  $[y_{n,1}]$  and  $[y_{n,1/2}]$  are in  $F^p$ , and the other named elements below the top row are in  $F^1$ .

For the second row of (4.1) we have an additive isomorphism

$$P(y_{n,0}) \otimes E(u_n) \cong T_p(y_{n,0}) \otimes E(w_{n,0}) \otimes P(y_{n,1}),$$

and the behavior of the first two factors is illustrated in the upper diagram of (8.3) below for  $j = 0$ , where  $\rho_n(0) = 1$ .

Using the notation of (2.6), we can consolidate the third and fourth rows of (4.1), and rewrite them as

$$EE_n \otimes PZ_n \rightsquigarrow \mathbf{Z}/p.$$

Thus we can rewrite (4.1) as

$$\begin{array}{ccc}
L_n \otimes T_{p^n}(z_{n+i} : i > 0) & \rightsquigarrow & L_n \otimes T_{p^n}(z_{n+i} : i > 0) \\
\otimes & & \otimes \\
T_p(y_{n,0}) \otimes E(w_{n,0}) & \rightsquigarrow & E(y_{n,1/2}) \\
\otimes & & \otimes \\
P(y_{n,1}) & \rightsquigarrow & P(y_{n,1}) \\
\otimes & & \otimes \\
EE_n \otimes PZ_n & \rightsquigarrow & \mathbf{Z}/p \\
\otimes & & \otimes \\
W_{n,0} & \rightsquigarrow & W_{n,0}.
\end{array}
\tag{4.2}$$

**Theorem 4.3.** For  $G_{*,*}^2$  as in (3.5), we have elements

$$\begin{array}{lll}
v_n \in G_{-2(p^n-1),1}^2, & y_{n,1} \in G_{2p,0}^2, & w_{n,i} \in G_{2p^{n+i}+1,0}^2, \\
y_{n,1/2} \in G_{2(p^n-1)+2p+1,0}^2, & \text{and } z_j \in G_{2(p^j+1),0}^2. &
\end{array}$$

The  $E_2$  term of the odd primary Adams spectral sequence for  $k(n)^*(K_2)$  is

$$P(v_n) \otimes L_n \otimes P(y_{n,1}) \otimes E(y_{n,1/2}) \otimes W_{n,0} \otimes TZ_{n,0}$$

plus  $S_{n,0} \otimes M_{n,0}$  from §2.3, the elements annihilated by  $v_n$ .

*Proof.* The  $Q_n$  homology of  $H^*K_2$  gives us the trivial  $E(Q_n)$ -module part. The rest is free over  $E(Q_n)$ . The Ext groups for both kinds of modules are as in (3.7). The result follows.  $\square$

## 5. ILLUSTRATION FOR $n = 2$

In this section we will sometimes write our generators other than  $\iota_2$  with subscripts enclosed in parentheses indicating their dimensions. In the cohomological case we have  $|v_2| = 2 - 2p^2$ , which is  $-16$  for  $p = 3$ .

Thus we have

$$\begin{aligned}
H^*K_2 &= P(\iota_2) \otimes E(u_s : s \geq 0) \otimes P(z_s : s > 0) \\
&= P(\iota_2) \otimes E(u_{(3)}, u_{(7)}, u_{(19)}, \dots) \otimes P(z_{(8)}, z_{(20)}, z_{(56)}, \dots) \quad \text{for } p = 3
\end{aligned}$$

with

$$(5.1) \quad \begin{aligned} Q_2(\iota_2) &= u_2, \\ Q_2(u_s) &= \begin{cases} z_2 & \text{for } s = 0 \\ z_1^p & \text{for } s = 1 \\ 0 & \text{for } s = 2 \\ z_{s-2}^{p^2} & \text{for } s \geq 3, \end{cases} \\ \text{and } Q_2(z_s) &= 0. \end{aligned}$$

The actions of  $Q_2$  on the first five  $u_s$ s imply that

$$\begin{aligned} Q_2(u_3 - z_1^{p^2-p}u_1) &= 0 \\ \text{and } Q_2(u_4 - z_2^{p^2-1}u_0) &= 0, \end{aligned}$$

so as in (2.2) we define

$$w_{2,0} := u_2, \quad w_{2,1} := u_3 - z_1^{p^2-p}u_1 \quad \text{and} \quad w_{2,2} := u_4 - z_2^{p^2-1}u_0,$$

with each being killed by  $Q_2$ .

The Adams  $E^1$ -term is

$$\begin{aligned} &P(v_2) \otimes P(\iota_2) \otimes E(u_s : s \geq 0) \otimes P(z_s : s \geq 1) \\ \text{with } &v_2 \in G_{2-2p^2,1}^2 \quad \iota_2 \in G_{2,0}^2 \\ &u_s \in G_{2p^s+1,0}^2 \quad z_s \in G_{2p^s+2,0}^2 \\ \text{and } &d^1 \iota_2 = v_2 u_2 \quad d^1 u_0 = v_2 z_2 \\ &d^1 u_1 = v_2 z_1^p \quad d^1 u_s = v_2 z_{s-2}^{p^2} \quad \text{for } s \geq 3. \end{aligned}$$

It follows that modulo  $v_2$ -torsion, the Adams  $E_2$ -term is

$$(5.2) \quad \begin{aligned} &P(v_2) \otimes P(\iota_2^p) \otimes E(\iota_2^{p-1}u_2, w_{2,1}, w_{2,2}) \otimes T_p(z_1) \otimes T_{p^2}(z_s : s \geq 3) \\ &= k(2)^* \otimes P(y_{2,1}) \otimes E(y_{2,1/2}, w_{2,1}, w_{2,2}) \otimes L_2 \otimes T_{p^2}(z_s : s \geq 3). \end{aligned}$$

**Lemma 5.3.** *For any prime and for all  $n$ , in the Adams spectral sequence for  $k(n)^*K_2$ ,*

- (i)  $z_s$  is a nontrivial permanent cycle for  $s > 0$ ,
- (ii) every power of  $\iota_2$  supports a differential, and
- (iii) some  $v_n$ -multiple of each  $z_s$  for  $s > n$  is killed by a differential.

Lemma 7.5 below is a similar statement.

*Proof.* (i) We know by [RW80, dual to Theorem 11.1] that

$$(5.4) \quad K(n)^*K_2 = K(n)^* \otimes \bigotimes_{0 < s < n} T_{p^{n-s}}(z_s).$$

This means that our Adams  $E_\infty$ -term must be congruent to

$$k(n)^* \otimes \bigotimes_{0 < s < n} T_{p^{n-s}}(z_s)$$

modulo  $v_n$ -torsion.

(ii) There is a fiber sequence

$$K(\mathbf{Z}, 2) \rightarrow K_2 \rightarrow K(\mathbf{Z}, 3)$$

for which the Serre spectral sequence collapses, that is

$$H^*K(\mathbf{Z}, 2) = P(\iota_2)$$

$$H^*K(\mathbf{Z}, 3) = E(u_s : s \geq 0) \otimes P(z_s : s \geq 1)$$

$$H^*K_2 = H^*K(\mathbf{Z}, 2) \otimes H^*K(\mathbf{Z}, 3).$$

This means that nothing in  $P(\iota_2)$  can be hit by an Adams differential for any  $n$ . Thus (5.4) implies that each power of  $\iota_2$  must support a nontrivial Adams differential. For  $n = 2$ , the action of  $Q_2$  in  $H^*K(\mathbf{Z}, 3)$  is given in (5.1).

(iii) We also have a  $p$ -local fiber sequence

$$K(\mathbf{Z}, 3) \rightarrow BP\langle 1 \rangle_{2p+2} \rightarrow BP\langle 1 \rangle_4$$

in which the second and third spaces have even dimensional cohomology. The generators  $z_i \in H^*K(\mathbf{Z}, 3)$  are in the image of the map from  $H^*BP\langle 1 \rangle_{2p+2}$ , so they map to permanent cycles in the Adams spectral sequences for both  $k(n)^*K(\mathbf{Z}, 3)$  and  $k(n)^*K_2$ .  $\square$

*It turns out that there is only one pattern of higher Adams differentials that leads to an answer meeting the conditions imposed by Lemma 5.3.*

For  $p = 3$ , it begins as follows.

$$(5.5) \quad \begin{array}{lll} [d^1(\iota_2) = v_2 w_{(19)} \in G_{3,1}^1] & d^2(\iota_2^2 w_{(19)}) = v_2^2 z_{(56)} \in G_{24,2}^2 & \\ d^3(\iota_2^3) = v_2^3 w_{(55)} \in G_{7,3}^3 & d^6(\iota_2^6 w_{(55)}) = v_2^6 z_{(164)} \in G_{68,6}^6 & \\ d^9(\iota_2^9) = v_2^9 w_{(163)} \in G_{19,9}^9 & d^{18}(\iota_2^{18} w_{(163)}) = v_2^{18} z_{(488)} \in G_{200,18}^{18} & \end{array}$$

**Remark 5.6. Standard notational abuse.** In (5.5) we are abusing notation for higher differentials in the usual way. For example, the source of the  $d^2$ , is written as  $\iota_2^2 u_{(19)}$ . However it is not really a product in  $G^2$  because  $\iota_2$  is no longer present there since it supported a  $d^1$ . Strictly speaking,  $\iota_2^2 u_{(19)}$  is an abbreviation for the Massey product

$$\langle v_2^2, u_{(19)}, u_{(19)}, u_{(19)} \rangle \in G_{23,0}^2.$$

Similarly  $\iota_2^3$  is code for

$$\langle u_{(19)}, v_2, v_2 u_{(19)}, v_2, u_{(19)} \rangle \in G_{6,0}^2 = G_{6,0}^3.$$

An introduction to Massey products can be found in [Rav86, A1.4] (and in [Rav04, A1.4]), which is an introduction to Peter May's definitive paper on the subject [May69].

Let

$$y_{2,s} := \iota_2^{p^s} \quad \text{for } s \geq 0 \quad \text{and} \quad y_{2,s+1/2} := y_{2,s}^{p-1} w_{2,s} \quad \text{for } s \geq 0.$$

Then for a general prime  $p$ , (5.5) reads

$$(5.7) \quad \begin{aligned} d^1(y_{2,0}) &= v_2 w_{2,0} & d^{p-1}(y_{2,1/2}) &= v_2^{p-1} z_3 \\ d^p(y_{2,1}) &= v_2^p w_{2,1} & d^{p^2-p}(y_{2,3/2}) &= v_2^{p^2-p} z_4 \\ d^{p^2}(y_{2,2}) &= v_2^{p^2} w_{2,2} & d^{p^3-p^2}(y_{2,5/2}) &= v_2^{p^3-p^2} z_5 \end{aligned}$$

The first differential reflects the fact that

$$Q_2 y_{2,0} = Q_2 \iota_2 = u_2 = w_{2,0} \in H^* K_2.$$

We also have

$$Q_2 u_s = \begin{cases} z_{2-s}^{p^s} & \text{for } 0 \leq s \leq 1 \\ 0 & \text{for } s = 2 \\ z_{s-2}^{p^2} & \text{for } s \geq 3. \end{cases}$$

These lead to

$$E_2 \cong T_p(z_1) \otimes P(v_2, y_{2,1}) \otimes E(y_{2,1/2}, w_{2,1}, w_{2,2}) \otimes T_{p^2}(z_{2+s} : s > 0)$$

modulo  $v_2$ -torsion as in (5.2), where

$$y_{2,1/2} = y_{2,0}^{p-1} w_{2,0}.$$

For  $p = 3$ , this reads

$$E_2 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(6)}) \otimes E(y_{(23)}, w_{(55)}, w_{(163)}) \otimes T_9(z_{(56)}, z_{(164)}, z_{(488)}, \dots).$$

Lemma 5.3(iii) requires a differential hitting a  $v_2$ -multiple of  $z_{(56)}$ . It cannot be supported by a  $v_2$ -torsion element since its target is torsion free. The only classes in low enough dimensions live in  $T_p(z_1) \otimes P(y_{2,1}) \otimes E(y_{2,1/2})$ . Since  $z_1$  is a permanent cycle, we can restrict our attention to  $P(y_{2,1}) \otimes E(y_{2,1/2})$ , which is  $P(y_{(6)}) \otimes E(y_{(23)})$  for  $p = 3$ . The only class in a dimension congruent to  $|z_3| - 1 \pmod{|v_2|}$  ( $55 \pmod{16}$  for  $p = 3$ ) is  $y_{2,1/2}$ , which is  $y_{(23)}$  for  $p = 3$ .

This gives us the second differential listed in (5.7). It also gives

$$E_p \cong T_p(z_1) \otimes P(v_2, y_{2,1}) \otimes E(w_{2,1}, w_{2,2}, w_{2,3}) \otimes T_{p^2}(z_{2+s} : s > 1)$$

modulo  $v_2$ -torsion, which for  $p = 3$  reads

$$E_3 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(6)}) \otimes E(w_{(55)}, w_{(163)}, w_{(471)}) \otimes T_9(z_{(164)}, z_{(488)}, z_{(1460)}, \dots).$$

What happens next? Something has to kill  $z_{(164)}$ , but none of the listed lower dimensional generators are in the right dimension to do so. However if  $d^3 y_{(6)} = v_2^3 w_{(55)}$ , we would get a new generator  $y_{2,3/2} = y_{2,1}^{p-1} w_{2,1}$ , which is  $y_{(67)} = y_{(6)}^2 w_{(55)}$  at  $p = 3$ . It is in the right dimension to kill  $v_2^6 z_{(164)}$ . Thus we get the next two differentials listed in (5.7) and we have (modulo  $v_2$ -torsion)

$$\begin{aligned} E_{p+1} &= E_{p^2-p} \cong T_p(z_1) \otimes P(v_2, y_{2,2}) \otimes E(y_{2,3/2}, w_{2,2}, w_{2,3}) \otimes T_{p^2}(z_{2+i} : i > 1) \\ \text{and} \quad E_{p^2-p+1} &\cong T_p(z_1) \otimes P(v_2, y_{2,2}) \otimes E(w_{2,2}, w_{2,3}, w_{2,4}) \otimes T_{p^2}(z_{2+i} : i > 2) \\ &\quad \text{with } w_{2,4} = y_{2,3/2} z_4^{p^2-1}. \end{aligned}$$

For  $p = 3$ , this reads

$$\begin{aligned} E_3 &= E_6 \cong T_3(z_{(8)}) \otimes P(v_2, y_{(18)}) \otimes E(y_{(67)}, w_{(163)}, w_{(471)}) \otimes T_9(z_{(164)}, z_{(488)}, \dots) \\ \text{and} \quad E_7 &\cong T_3(z_{(8)}) \otimes P(v_2, y_{(18)}) \otimes E(w_{(163)}, w_{(471)}, w_{(1379)}) \otimes T_9(z_{(488)}, z_{(1460)}, \dots). \end{aligned}$$

Note that at each stage we have the following factors:

- $k(2)^* \otimes T_p(z_1)$ ,
- the polynomial algebra generated by some  $p^k$ th power of  $\iota_2$ ,
- an exterior algebra on three generators (two  $w$ s plus a  $y$  or a third  $w$ ) with each having a dimension congruent to 3 or  $2p + 1 \pmod{|v_2|}$ , and
- a truncated polynomial algebra of height  $p^2$  on infinitely many  $z_i$ s having dimensions alternately congruent to 4 and  $2p + 2 \pmod{|v_2|}$ .

The exterior generator with the  $y$  label is the only one in a position to kill the next  $z$ . *These phenomena persist throughout the spectral sequence and generalize to larger values of  $n$ .* The factor  $T_p(z_1)$  generalizes to  $L_n$  as in (2.6). The dimension of each  $y_{n,i}$  is congruent to 2 modulo  $2p - 2$ , while those of the exterior generators and the  $z_i$ s are congruent to 3 and 4 respectively. Half the generators remove  $y_{n,i}$  and  $w_{n,i}$ , replacing them with  $y_{n,i+1}$  and  $y_{n,i+1/2}$ . The others remove  $z_{n+i+1}$  and replace  $y_{n,i+1/2}$  by  $w_{n,i+n+1}$ .

We want to extend (5.7) further with differentials supported by higher powers of  $\iota_2$  in the left column and ones killing  $v_2$ -multiples of higher  $z_i$ s in the right column.

Since  $v_2 z_3^{p^2} = 0$  in  $G^2$ , the element

$$w_{2,3} := y_{2,0}^{p-1} w_{2,0} z_3^{p^2-1} = \langle v_2^{p-2}, v_2 z_3, z_3^{p^2-1} \rangle \quad \text{as in (2.2)}$$

is a  $d^{p-1}$ -cycle and hence a target for  $y_{2,3}$ . Thus we have

$$d^{p^3-p+2} y_{2,3} = v_2^{p^3-p+2} w_{2,3} \quad \text{and} \quad d^{p^4-p^3+p-2} (y_{2,3}^{p-1} w_{2,3}) = v_2^{p^4-p^3+p-2} z_6$$



with

$$y_{2,7/2} := y_{2,3}^{p-1} w_{2,3} = \langle v_2^{(p-1)(p^3-1)}, \underbrace{w_{2,3}, \dots, w_{2,3}}_{p \text{ factors}} \rangle.$$

We denote the indices of the differentials on  $y_{2,i}$  and  $y_{2,i+1/2}$  by  $\rho_2(i)$  and  $\rho_2(i + 1/2)$ . Hence the  $i$ th row of (5.7) is

$$d^{\rho_2(i)} y_{2,i} = v_2^{\rho_2(i)} w_{2,i} \quad \text{and} \quad d^{\rho_2(i+1/2)} y_{2,i+1/2} = v_2^{\rho_2(i+1/2)} z_{i+n+1},$$

where

$$w_{2,i} = \begin{cases} u_2 & \text{for } i = 0 \\ u_3 - z_1^{p^2-p} u_1 & \text{for } i = 1 \\ u_4 - z_2^{p^2-1} u_0 & \text{for } i = 2 \\ y_{2,i-3}^{p-1} w_{i-1} z_i^{p^2-1} = y_{2,i-5/2} z_i^{p^2-1} & \text{for } i \geq 3 \end{cases}$$

as in (2.2).

The following is a special case of Lemma 6.4 below.

**Proposition 5.8.** *The indices  $\rho_2(i)$  and  $\rho_2(i + 1/2)$  for integers  $i \geq 0$  are*

$$\rho_2(i) = \begin{cases} p^i & \text{for } 0 \leq i \leq 2 \\ p^i - p^{i-2} + 1 + \rho_2(i-3) & \text{for } i \geq 3 \end{cases}$$

and  $\rho_2(i + 1/2) = p^{i+1} - \rho_2(i)$ .

## 6. NUMBERS AND DEFINITIONS

In this section we give some definitions and compute some numbers we need. We already have elements  $y_{n,i}$ ,  $z_i$  and  $w_{n,i}$  and  $y_{n,i+1/2}$  with

$$(6.1) \quad \begin{aligned} |y_{n,i}| &= 2p^i, \\ |z_i| &= 2(p^i + 1), \\ |y_{n,i+1/2}| &= |y_{n,i}^{p-1} w_{n,i}| = 2p^i(p-1) + |w_{n,i}|, \\ \text{and } |w_{n,i+n+1}| &= |y_{n,i+1/2} z_{n+i+1}^{p^n-1}| = |y_{n,i}^{p-1} w_{n,i} z_{n+i+1}^{p^n-1}| \\ &= 2p^i(p-1) + |w_{n,i}| + 2(p^n-1)(p^{n+i+1} + 1) \\ &= 2p^i(p^{2n+1} - p^{n+1} + p - 1) + 2(p^n-1) + |w_{n,i}| \\ &= 2p^i(p^{n+1} c_n + c_1) + 2c_n + |w_{n,i}|, \end{aligned}$$

where  $c_k := p^k - 1$ .

Regarding these as functions of  $i$ , we will see that each one satisfies a recursive formula similar to that for  $|w_{n,i}|$ . To study such functions, we need some notation.

**Definition 6.2.**

- (i) For a fixed positive integer  $n$ , let  $i_1 = \lfloor i/(n+1) \rfloor$  and  $i_0 = i - (n+1)i_1$  (the reduction of  $i$  modulo  $n+1$ ) for any integer  $i$ .
- (ii) Similarly let  $i'_1 = \lfloor i/n \rfloor$  and  $i'_0 = i - ni'_1$ .
- (iii) Let

$$g_n(i) := \frac{p^i - p^{i_0}}{p^{n+1} - 1} = \begin{cases} 0 & \text{for } 0 \leq i \leq n \\ p^{i-n-1} + g_n(i-n-1) & \text{for } i \geq n+1 \end{cases}$$

$$= p^{i-(n+1)} + p^{i-2(n+1)} + p^{i-3(n+1)} + \dots + p^{i_0}.$$

It follows that  $i$  can be written uniquely as

$$i = \begin{cases} i_0 + (n+1)i_1 & \text{with } 0 \leq i_0 \leq n \\ i'_0 + ni'_1 & \text{with } 0 \leq i'_0 < n \end{cases}$$

**Lemma 6.3.** For a fixed positive integer  $n$ , suppose we have an integer valued function  $f_n(i)$  defined for integers  $i \geq 0$  and satisfying the recursive equation

$$f_n(i+n+1) = ap^i + f_n(i) + b \quad \text{for constants } a \text{ and } b.$$

Then, with notation as in Definition 6.2,

$$f_n(i) = ag_n(i) + b\lfloor i/(n+1) \rfloor + f_n(i_0) = ag_n(i) + f_n(i_0) + bi_1$$

$$\equiv a \left( \frac{p^{i_0} - p^{i_0}}{p-1} \right) + f_n(i_0) + b\lfloor i/(n+1) \rfloor \pmod{(p^n - 1)},$$

and the latter expression is an integer.

*Proof.* Iterating the recursion relation gives

$$\begin{aligned} f_n(i) &= ap^{i-(n+1)} + f_n(i - (n+1)) + b \\ &= a(p^{i-(n+1)} + p^{i-2(n+1)}) + f_n(i - 2(n+1)) + 2b \\ &\vdots \\ &= a(p^{i-(n+1)} + p^{i-2(n+1)} + \dots + p^{i_0}) + f_n(i_0) + i_1b \\ &= ag_n(i) + f_n(i_0) + \lfloor i/(n+1) \rfloor. \end{aligned}$$

The congruence modulo  $(p^n - 1)$  follows from the fact that  $p^n \equiv 1$ . □

**Lemma 6.4.** The values of  $a$ ,  $b$ ,  $f_n(i_0)$  and  $f_n(i) \pmod{2(p^n - 1)}$  for some functions of interest are shown in the following table, where again  $c_k := p^k - 1$ .

$f_n(i)$	$a$	$b$	$f_n(i_0)$	$f_n(i) \bmod 2c_n$
$g_n(i)$	1	0	0	
$\lfloor i/(n+1) \rfloor$	0	1	0	
$ y_{n,i}  = 2p^i$	$2c_{n+1}$	0	$2p^{i_0}$	$2p^{i_0}$
$ u_i  = 2p^i + 1$	$2c_{n+1}$	0	$2p^{i_0} + 1$	$2p^{i_0} + 1$
$ z_i  = 2p^i + 2$	$2c_{n+1}$	0	$2p^{i_0} + 2$	$2p^{i_0} + 2$
$\rho_n(i)$	$pc_n$	1	$p^{i_0}$	
$\rho_n(i + 1/2)$	$p^{n+1}c_1$	-1	$p^{i_0}c_1$	
$ w_{n,i} $	$2(c_1 + p^{n+1}c_n)$	$2c_n$	$2p^{n+i_0} + 1$	$2p^{i_0} + 1$
$ y_{n,i+1/2} $	$2p^{n+1}(c_1 + c_n)$	$2c_n$	$2p^{i_0}(c_n + p) + 1$	$2p^{i_0+1} + 1$

In particular there are relations

$$(6.5) \quad \begin{aligned} |w_{n,i}| &\leq |u_{n+i}| = 2p^{n+i} + 1 \\ \rho_n(i + 1/2) + \rho_n(i) &= p^{i+1}, \\ \text{and } \rho_n(i) &\leq p^i < \rho_n(i + 1). \end{aligned}$$

More explicitly for integers  $i \geq 0$ ,

$$(6.6) \quad \begin{aligned} \rho_n(i) &= (p^{n+1} - p)(p^{i-(n+1)} + p^{i-2(n+1)} + \dots + p^{i_0}) \\ &\quad + p^{i_0} + i_1 \\ &= p^i - p^{i-n} + p^{i-n-1} - p^{i-2n-1} + \dots \\ &\quad + p^{i_0+n+1} - p^{1+i_0} + p^{i_0} + i_1 \\ &= \begin{cases} p^i & \text{for } 0 \leq i \leq n \\ p^i - p^{i-n} + p^{i-n-1} + 1 & \text{for } n+1 \leq i \leq 2n+1 \\ p^i - p^{i-n} + p^{i-n-1} - p^{i-2n-1} + p^{i-2n-2} + 2 & \text{for } 2n+2 \leq i \leq 3n+2 \\ \vdots & \end{cases} \end{aligned}$$

$$\text{and } \rho_n(i + 1/2) = p^{i+1} - \rho_n(i).$$

We do not need the values of  $f_n(i) \bmod 2(p^n - 1)$  in the cases where it is not shown.

*Proof of Lemma 6.4.* The first five functions are defined explicitly, so filling in the columns for them is straightforward. We also know the values  $f_n(i_0)$  for the last four functions listed, so it remains to determine the constants  $a$  and  $b$  for each of them. The congruences module  $2(p^n - 1)$  are also straightforward.

The constants  $a$  and  $b$  for  $\rho_2(i)$  were given in Proposition 5.8.

Our differentials

$$d^{\rho_n(i)}(y_{n,i}) = v_n^{\rho_n(i)} w_{n,i} \quad \text{and} \quad d^{\rho_n(i+1/2)}(y_{n,i+1/2}) = v_n^{\rho_n(i+1/2)} z_{n+i+1}.$$

imply

$$(6.7) \quad \begin{aligned} & |y_{n,i}| + 1 + 2(p^n - 1)\rho_n(i) = |w_{n,i}| \\ \text{and} \quad & |y_{n,i+1/2}| + 1 + 2(p^n - 1)\rho_n(i + 1/2) = |z_{i+n+1}|. \end{aligned}$$

This means that the constants for  $\rho_n(i)$  and  $\rho_n(i + 1/2)$  are determined by those of  $|y_{n,i}|$  and  $|z_i|$ , which are known, and those of  $|w_{n,i}|$  and  $|y_{n,i+1/2}|$ , to which we now turn.

The constants  $a$  and  $b$  for  $|w_{n,i}|$  are given by (6.1).

For  $|y_{n,i+1/2}|$ , (2.3) implies

$$\begin{aligned} y_{n,i+1/2+n+1} &= y_{n,i+n+1}^{p-1} w_{n,i+n+1} = y_{n,i+n+1}^{p-1} w_{n,i} y_{n,i}^{p-1} z_{i+n+1}^{p^n-1} \\ &= y_{n,i+n+1}^{p-1} y_{n,i+1/2} z_{i+n+1}^{p^n-1}, \\ \text{so} \quad |y_{n,i+1/2+n+1}| &= |y_{n,i+1/2}| + |y_{n,i+n+1}^{p-1} z_{i+n+1}^{p^n-1}| \\ &= |y_{n,i+1/2}| + 2(p-1)p^{i+n+1} + 2(p^n - 1)(p^{i+n+1} + 1), \end{aligned}$$

which gives the stated values of  $a$  and  $b$ .  $\square$

## 7. PRELIMINARIES BEFORE THE PROOF

Before proving Theorems 1.1 and 1.2, we have the following observation.

**Proposition 7.1. Divisibility Criterion.** *If in the Adams spectral sequence for  $k(n)^* X$ ,  $d^r(\alpha) = v_n^r \beta$ , then*

$$|\alpha| + 1 + 2r(p^n - 1) = |\beta|,$$

*i.e.,  $|\beta|$  is congruent to  $1 + |\alpha|$  modulo  $|v_n|$ .*

*Proof of Theorem 1.1.* In [Laz01, Corollary 11.8] and [Rob89, Theorem 2.3], the odd primary  $k(n)$  is shown to be  $A_\infty$ . In private communication, Lazarev says that his argument for  $k(n)$  works just as well for  $p = 2$ .

In [Rob87, p. 257], Robinson produces a Universal Coefficient Theorem for  $A_\infty$  spectra. In our case this gives the spectral sequence of Theorem 1.1. For spaces of finite type with  $K(n)_*(X)$  finitely generated,  $k(n)_*(X)$  is the sum of a free module (of finite dimension) over  $k(n)_*$  and a sum of torsion modules,  $T_k(v_n)$ . The above Ext is easy to compute and everything is in  $\text{Ext}^0$  and  $\text{Ext}^1$ . More precisely:

$$\text{Ext}_{k(n)_*}^{0,*}(k(n)_*, k(n)_*) = k(n)^*$$

$$\text{Ext}_{k(n)_*}^{1,*}(T_k(v_n), k(n)_*) = T_k(v_n)$$

with generator in  $\text{Ext}_{k(n)_*}^{1, |v_n^k|}$  and  $v_n \in \text{Ext}_{k(n)_*}^{0, -2(p^n-1)}$

The entire  $E_2$  term is in  $\text{Ext}^0$  and  $\text{Ext}^1$ . This is peculiar to  $k(n)$ . As a result, the spectral sequence collapses.  $\square$

*Proof of Theorem 1.2.* If we have  $d^r(\alpha) = v_n^r \beta$  in the Adams spectral sequence for  $k(n)^*(X)$ , it means we have (a cohomology)  $T_r(v_n)$  with generator in the degree of  $\beta$ . From the UCT, to get this, we must have a (homology)  $T_r(v_n)$  with generator in the degree of  $\alpha$ . To get this in the Adams spectral sequence for  $k(n)_*(X)$ , we must have a differential  $d_{\rho_n}(\beta') = v_n^r \alpha'$  with the mentioned degrees. Reverse the argument to get the other direction.  $\square$

**Remark 7.2.** There is a way to invert  $v_n$  in the Adams spectral sequence which converts it to a full plane spectral sequence, the *localized* Adams spectral sequence, rather than an upper half plane one. Details can be found in [MRS01, §2.3].

**Remark 7.3.** It seems likely that Theorem 1.2 also follows from the method of synthetic spectra of Piotr Pstrągowski [Pst23], but we prove it with more prosaic methods. We leave the synthetic approach to the interested reader.

Before we state the next result, we need

$$(7.4) \quad H_*K_2 = \Gamma(\iota_2^*) \otimes \Gamma(z_i^* : i > 0) \otimes E(u_i^* : i \geq 0).$$

Here we have  $y_{n,j}^* = \gamma_{p^j}(\iota_2^*)$  dual to  $y_{n,j}$  in cohomology.

In Theorem 9.4(i), we compute the  $E_2$  term for the Adams spectral sequence for  $k(n)_*(K_2)$ . In particular,  $\Gamma(y_{n,1}^*)$  is there.

The following is a refined version of Lemma 5.3. Unfortunately, the proof of the crucial refinement is intertwined with the proof of the computation of the ASS, Theorem 8.1, in the next section. This result seems to best fit this section and it is easy enough to read of what is needed from Theorem 8.1.

**Lemma 7.5.** *For any prime, the  $z_i$  are all permanent cycles in the Adams spectral sequence for  $k(n)^*(K_2)$  and there is a non-zero differential  $d^r(y_{n,i})$  for some  $r \leq p^i$ . In the Adams spectral sequence for  $k(n)_*(K_2)$ ,  $v_n^r y_{n,i}^*$  is hit by a differential for some  $r \leq p^i$ .*

*Proof.* The image of the map

$$BP^*(K(\mathbf{Z}/p, m)) \rightarrow H^*K(\mathbf{Z}/p, m)$$

is computed by Tamanoi in [Tam97] (and much earlier in his 1983 masters thesis in Japan) and then again later in [RWY98]. In particular, the answer for  $m = 2$  contains the  $z_i$ , where  $i > 0$ . This map factors through  $k(n)^*(K_2)$  so we conclude that the  $z_i$  cannot support a differential.

Let  $b_\ell \in k(n)_{2\ell}\mathbb{C}P^\infty$  be the standard generator and consider the composition

$$\mathbb{C}P^\infty \xrightarrow{p} \mathbb{C}P^\infty \longrightarrow K_2.$$

Define  $b(s) = \sum_\ell b_\ell s^\ell$  and  $b_{(i)} = b_{p^i}$ . Note that  $b_{(i)}$  maps to  $y_{n,i}^* \in k(n)_*(K_2)$ . We follow [RW77, Theorem 3.8(ii)] and use the fact that for  $k(n)$ ,  $[p](s) = v_n s^{p^n}$ . The composition above takes  $b(s)$  to zero, but the first map takes  $b(s) \rightarrow b(v_n s^{p^n})$ . In particular, we see that  $v_n^{p^i} b_{(i)}$  maps to zero, giving  $v_n^{p^i} y_{n,i}^* = 0 \in k(n)_*(K_2)$ .

Since we must have a  $d_{\rho_n}(\beta) = v_n^r y_{n,i}^*$  with  $r \leq p^i$ , from Theorem 1.2, we must have a corresponding  $d_n^\rho(\alpha') = v_n^r \beta'$  with  $|\alpha'| = 2p^i = |y_{n,i}^*|$ . We want to show that  $y_{n,i} = \alpha'$ . We give the proof for odd primes,  $p = 2$  requires modifications. To do this, we use induction on  $i$ . We can begin the induction with  $i = 0$ , where  $d^1(y_{n,0}) = v_n u_n$  from (2.4). We now assume the result for  $y_{n,i-1}$ . More than that, we assume Theorem 8.1 to the point where we have computed  $d^{\rho_n(i-1)}(y_{n,i-1})$  and obtained  $E_{1+\rho_n(i-1)}$ . We will be finished if we can show that the only element of  $E_{1+\rho_n(i-1)}$  that can have a differential on it in degree  $2p^i$  is  $y_{n,i}$ . The only elements that can have differentials are in the  $k(n)^*$ -free part of  $E_{1+\rho_n(i-1)}$ . In Theorem 8.1, we have  $\ell = 2(i-1)$ , so the  $k(n)^*$ -free part of  $E_{1+\rho_n(i-1)}$  is easy to read off as

$$E(y_{n,i-1/2}) \otimes T_{p^n}(z_{n+i}) \otimes P(y_{n,i}) \otimes W_{n,i-1} \otimes L_n \otimes TZ_{n,i}$$

The elements of  $L_n$  cannot be used because they give us  $K(n)^*(K_2)$ . The lowest degree element of  $TZ_{n,i}$  is  $z_{n+i+1}$  and its degree is higher than  $2p^i$  so we can ignore  $TZ_{n,i}$ . The degree of  $z_{n+i}$  of  $T_{p^n}(z_{n+i})$  is also too high. All we have left to eliminate is  $E(y_{n,i-1/2}) \otimes W_{n,i-1}$ . The element of lowest degree in  $W_{n,i-1}$  is  $w_{n,i}$ . For  $i \leq n$ , the degree of  $w_{n,i}$  is  $2p^{n+i} + 1$  by (2.2) and too big to consider. When  $i > n$ , we rely on Lemma 6.4. Here we have the degree of  $w_{n,i}$  is given as

$$2(c_1 + p^{n+1}c_n)p^{i-n-1} + 2c_n + |w_{n,i-n-1}|.$$

The term  $2p^{n+1}c_n p^{i-n-1} = 2(p^n - 1)p^i$ , so the degree is too high to worry about. All that is left is  $y_{n,i-1/2}$ , but it has odd degree. Although unnecessary at this stage, the degree of  $y_{n,i-1/2}$  is also greater than  $2p^i$ .  $\square$

## 8. THE ADAMS SPECTRAL SEQUENCE FOR ODD PRIMES

The  $E_2$ -term of the odd primary Adams spectral sequence for  $k(n)^*(K_2)$  is the subject of Theorem 4.3.

**Theorem 8.1. Adams differentials and intermediate terms for  $k(n)^*(K_2)$ .**

(i) In the odd primary Adams spectral sequence for  $k(n)^*(K_2)$ , the differentials  $d^r$  for  $r \geq 1$  are

$$d^1(y_{n,0}) = v_n u_n$$

$$d^1(u_s) = \begin{cases} v_n z_{n-s}^{p^s} & \text{for } 0 \leq s < n \\ v_n z_{s-n}^{p^n} & \text{for } s > 2n, \end{cases}$$

$$d^{\rho_n(i+1/2)}(y_{n,i+1/2}) = v_n^{\rho_n(i+1/2)} z_{n+i+1} \quad \text{for } i \geq 0$$

and  $d^{\rho_n(i)}(y_{n,i}) = v_n^{\rho_n(i)} w_{n,i} \quad \text{for } i > 0,$

where  $\rho_n(\ell/2)$  is as in (6.6).

(ii) For each  $\ell \geq 0$ ,

$$\begin{aligned} E_{1+\rho_n(\ell/2)} &= E_{\rho_n((\ell+1)/2)} \\ &= \bigoplus_{0 \leq k \leq \ell} (S_{n,k/2} \otimes M_{n,k/2}) \\ &\quad \oplus \left( k(n)^* \otimes \left\{ \begin{array}{l} E(y_{n,(\ell+1)/2}) \otimes T_{p^n}(z_{n+1+\ell/2}) \\ \text{for } \ell \text{ even} \\ T_p(y_{n,(\ell+1)/2}) \otimes E(w_{n,(\ell+1)/2}) \\ \text{for } \ell \text{ odd} \end{array} \right\} \otimes M_{n,(\ell+1)/2} \right) \end{aligned}$$

for  $M_{n,\ell/2}$  and  $S_{n,\ell/2}$  as in (2.11).

In (ii) note that as  $\ell$  goes to  $\infty$ , both the expressions enclosed in braces go to  $\mathbf{Z}/p$  and  $M_{n,(\ell+1)/2}$  goes to  $L_n$ . Hence the last summand goes to  $k(n)^* \otimes L_n$ , and (ii) implies that the Adams  $E_\infty$ -term is the  $k(n)^*$ -module described in Theorem 2.12.

*Proof of Theorem 2.12 and Theorem 8.1(ii) assuming Theorem 8.1(i).* The Adams  $E_1$ -term is

$$k(n)^* \otimes H^*K_2.$$

Our  $d^1$  for an odd prime  $p$  was computed in §2.3 and our  $E_2$  in Theorem 4.3. The remaining  $k(n)^*$ -free part was  $k(n)^* \otimes E(y_{n,1/2}) \otimes M_{n,0}$ , but  $M_{n,0} = T_{p^n}(z_{n+1}) \otimes M_{n,1/2}$ , giving us the answer for the above Adams  $E_2$ -term.

Higher differentials involve multiplication by higher powers of  $v_n$ , so they cannot effect the torsion submodule.

The remaining Adams differentials, starting with  $d^{\rho_n(1/2)}(y_{n,1/2}) = v_n^{\rho_n(1/2)} z_{n+1}$  (where  $\rho_n(1/2) = p - 1$ ), have the following effects on the indicated subquotient

rings of  $E_2$ ,  $E_{1+\rho_n(i)}$  and  $E_{1+\rho_n(i+1/2)}$  for integers  $i \geq 0$ .

$$\begin{aligned}
& y_{n,i+1/2} \mapsto v_n^{\rho_n(i+1/2)} z_{n+i+1} \\
& k(n)^* \otimes E(y_{n,i/2}) \otimes T_{p^n}(z_{n+i+1}) \sim (k(n)^* \otimes E(w_{n,n+i+1})) \\
& \quad \oplus (T_{\rho_n(i+1/2)}(v_n) \otimes z_{n+i+1} T_{p^{n-1}}(z_{n+i+1})) \\
& = (k(n)^* \otimes E(w_{n,n+i+1})) \oplus S_{n,i+1/2} \\
(8.2) \quad & y_{n,i+1} \mapsto v_n^{\rho_n(i+1)} w_{n,i+1} \\
& k(n)^* \otimes T_p(y_{n,i+1}) \otimes E(w_{n,i+1}) \sim (k(n)^* \otimes E(y_{n,i+3/2})) \\
& \quad \oplus (T_{\rho_n(i+1)}(v_n) \otimes w_{n,i+1} T_{p-1}(y_{n,i+1})) \\
& = (k(n)^* \otimes E(y_{n,i+3/2})) \oplus S_{n,i+1}
\end{aligned}$$

These are illustrated by the following diagrams, in which an arrow  $\alpha \rightarrow \beta$  labeled by  $v_n^r$  for some  $r$  means that  $d^r \alpha = v_n^r \beta$ . Within each of the two diagrams, all arrows should bear the same label, but all but two labels have been omitted to avoid clutter.

$$\begin{array}{ccccccc}
1 & & y_{n,i} & & y_{n,i}^2 & \cdots & y_{n,i}^{p-2} & & y_{n,i}^{p-1} \\
v_n^{\rho_n(i)} \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow v_n^{\rho_n(i)} \\
w_{n,i} & & w_{n,i} y_{n,i} & & w_{n,i} y_{n,i}^2 & \cdots & w_{n,i} y_{n,i}^{p-2} & & w_{n,i} y_{n,i}^{p-1} =: y_{n,i+1/2}
\end{array}$$

(8.3)

$$\begin{array}{ccccccc}
y_{n,i+1/2} & & y_{n,i+1/2} z_{n+i+1} & & \cdots & & y_{n,i+1/2} z_{n+i+1}^{p^n-2} & & y_{n,i+1/2} z_{n+i+1}^{p^n-1} =: w_{n,i+n+1} \\
v_n^{\rho_n(i+1/2)} \swarrow & & \swarrow & & \swarrow & & \swarrow & & \swarrow v_n^{\rho_n(i+1/2)} \\
1 & & z_{n+i+1} & & \cdots & & z_{n+i+1}^{p^n-2} & & z_{n+i+1}^{p^n-1}
\end{array}$$

Let  $S_{n,i}$  and  $S_{n,i+1/2}$  be as in (2.11). Then the new torsion modules created by  $d^{\rho_n(i)}$  for  $i > 0$  and  $d^{\rho_n(i+1/2)}$  for  $i \geq 0$  are respectively

$$S_{n,i} \otimes M_{n,i} \quad \text{and} \quad S_{n,i+1/2} \otimes M_{n,i+1/2}.$$

These give Theorem 8.1(ii).

This means that the Adams  $E_\infty$ -term has the form indicated in Theorem 2.12. We have to be sure that there are no nontrivial extensions in  $k(n)^*$ -module structure.



Suppose that for some  $i$ ,  $v_n^{\rho_n(i)} w_{n,i}$  is not zero but instead has higher filtration than expected. This would mean

$$v_n^{\rho_n(i)} w_{n,i} = v_n^{r_{n,i}} x_{n,i} \quad \text{for some } x_{n,i} \text{ with } r_{n,i} > \rho_n(i).$$

Then we would have

$$v_n^{\rho_n(i)} (w_{n,i} - v_n^{r_{n,i} - \rho_n(i)} x_{n,i}) = 0,$$

and we could define

$$w'_{n,i} := w_{n,i} - v_n^{r_{n,i} - \rho_n(i)} x_{n,i} \quad \text{with } v_n^{\rho_n(i)} w'_{n,i} = 0.$$

Of course this could also be in a higher filtration, but this has to end because each degree of Theorem 2.12 is finite. This follows from  $K(n)^*(K_2)$  finite over  $K(n)^*$  and  $K_2$  being of finite type. In the end, our final element would represent the same element in  $E_\infty$  as  $w_{n,i}$ , so the  $k(n)^*$ -module structure would still be as stated in Theorem 2.12.

A similar argument works for the relation  $v_n^{\rho_n(i+1/2)} z_{n,i+1} = 0$ .  $\square$

*Proof of Theorem 8.1(i).* Our proof starts with showing the asserted differentials must happen. Then we have to show that there are no additional differentials. This is where the full power of *The Pairing* comes in.

Assume by induction that we have  $E_{\rho_n(i-1/2)+1}$ . We must have a  $d^r(y_{n,i}) = v_n^r q$  where  $q$  has odd degree and  $r \leq p^i$  by Lemma 7.5. There are few odd degree elements in this range. We will show that if  $q = w_{n,i+1}$ , we would have  $r > p^i$ . This eliminates all  $q = w_{n,i+j}$ ,  $j > 1$ , because their degree is even higher. We want to show

$$|w_{n,i+1}| - 1 - |y_{n,i}| > 2p^i(p^n - 1)$$

Using the formula for  $w_{n,i+1}$  of (6.7), the left hand side becomes

$$\begin{aligned} & |y_{n,i+1}| + 1 + 2\rho_n(i+1)(p^n - 1) - 1 - |y_{n,i}| \\ &= 2(p-1)p^i + 2\rho_n(i+1)(p^n - 1), \end{aligned}$$

which makes the desired inequality

$$\rho_n(i+1) > \frac{p^{i+n} - p^{i+1}}{p^n - 1} = \frac{p^i(p^n - p)}{p^n - 1}$$

It is enough to have  $\rho_n(i+1) > p^i$ , because this is larger than the term on the right, but this is in (6.5).

The only remaining elements of odd degree that have degree less than  $|y_{n,i}| + 2p^i(p^n - 1)$  are  $y_{n,i}^s w_{n,i}$  for  $s > 0$ . However, since we know  $w_{n,i}$  meets the *Divisibility Criterion*, we must have  $s$  at least  $p^n - 1$ . Then the index of the differential would be  $p^i + \rho_n(i)$ , and this is greater than  $p^i$  so can't happen. We conclude that we *must* have  $d^{\rho_n(i)} y_{n,i} = v_n^{\rho_n(i)} w_{n,i}$  as claimed.

The action of this differential takes place in  $k(n)^* \otimes P(y_{n,i}) \otimes E(w_{n,i})$  which can be broken up as  $k(n)^* \otimes P(y_{n,i+1}) \otimes T_p(y_{n,i}) \otimes E(w_{n,i})$ , as we saw in (8.3). The remaining  $v_n$ -torsion free part is  $k(n)^* \otimes P(y_{n,i+1}) \otimes E(y_{n,i+1/2})$ , giving us  $E_{\rho_n(i)+1}$ .

We already know the  $P(v_n)$ -free elements that survive to  $E_\infty$ . Since  $z_{n+i+1}$  is not one of them but is a permanent cycle, we know that some  $v_n^r z_{n+i+1}$  must be hit by a differential coming from an odd degree element.

**Lemma 8.4.** *If  $d^r(w_{n,i+j}) = v_n^r z_{n+i+1}$  for some  $j > 0$ , then  $r \leq \rho_n(i - 1/2)$ .*

*Proof.* It is enough to study the  $j = 1$  case. If this differential is too short, then it is even shorter for  $j > 1$ . We would have

$$|w_{n,i+1}| + 1 + 2r(p^n - 1) = |z_{n+i+1}|.$$

Replace  $|w_{n,i+1}|$  using (6.7)

$$|y_{n,i+1}| + 1 + 2\rho_n(i+1)(p^n - 1) + 1 + 2r(p^n - 1) = |z_{n+i+1}|.$$

Plugging in the numbers for  $y_{n,i+1}$  and  $z_{n+i+1}$  and rearranging, we get

$$2\rho_n(i+1)(p^n - 1) + 2r(p^n - 1) = 2p^{n+i+1} - 2p^{i+1} = 2p^{i+1}(p^n - 1).$$

So,  $r = p^{i+1} - \rho_n(i+1)$ .

We need to show this is  $\leq \rho_n(i - 1/2)$ . We use the formulas from Lemma 6.4. We need to show that  $p^{i+1} \leq \rho_n(i+1) + \rho_n(i - 1/2)$ . This is easy for small  $i$ , so using the formulas and induction, we need to show

$$p^{i+n+2} \leq \rho_n(i+n+2) + \rho_n(i+n+1/2).$$

The right hand side is

$$p^{i+2}(p^n - 1) + 1 + \rho_n(i+1) + p^{n+i}(p-1) - 1 + \rho_n(i - 1/2).$$

Expanding and using induction, this is greater than or equal to

$$p^{n+i+2} - p^{i+2} + p^{n+i+1} - p^{n+i} + p^{i+1}.$$

For this to be greater than or equal to  $p^{n+i+2}$ , we need

$$p^{n+i+1} + p^{i+1} \geq p^{n+i} + p^{i+2}.$$

This is obvious for  $n > 1$ , and we get an equality when  $n = 1$ . □

Lemma 8.4 rules out all  $w_{n,i+j}$  with  $j > 0$  as the source of a differential hitting a  $v_n$ -multiple of  $z_{n+i+1}$  because we assume that  $E_{1+\rho_n(i-1/2)}$  has already been computed.

Now the only odd degree elements left in degrees less than  $|z_{n+i+1}|$  are the  $y_{n,i+1}^s y_{n,i+1/2}$ . We know  $y_{n,i+1/2}$  would work with differential  $\rho_n(i+1/2)$  because of the *Divisibility Criterion*. The *Divisibility Criterion* requires  $s$  to be a multiple of  $p^n - 1$ . The lowest non-zero  $s$  is  $s = p^n - 1$  and this would give a differential of

length  $\rho_n(i + 1/2) - p^{i+1}$ , but  $p^{i+1} > \rho_n(i + 1/2)$  so this cannot happen. We *must* have  $d^{\rho_n(i+1/2)}(y_{n,i+1/2}) = v_n^{\rho_n(i+1/2)} z_{n+i+1}$ .

As we saw in (8.3), the action of  $d^{\rho_n(i+1/2)}$  takes place in  $P(v_n) \otimes E(y_{n,i+1/2}) \otimes T_{p^n}(z_{n+i+1})$  and results in the  $P(v_n)$ -free part being  $E(w_{n,i+(n+1)})$ , giving us  $E_{\rho_n(i+1/2)+1}$ .

Having computed these differentials, we can use *The Pairing* of Theorem 1.2 to get the dual differentials for the Adams spectral sequence for  $k(n)_*(K_2)$  in Theorem 9.4. We first show  $d_{\rho_n(i)}(w_{n+i}^*) = v_n^{\rho_n(i)} y_i^*$ . We know, from Lemma 7.5 that some differential must hit some  $v_n^r y_i^*$  with  $r \leq p^i$ . From *The Pairing*, we know that some element,  $q$ , in the degree of  $y_i^*$  must have  $d_{\rho_n(i)}(q) = v_n^{\rho_n(i)} q$ . However, in  $E_{\rho_n(i-1/2)}$ , we see that  $y_i^*$  is the only element there is in that degree and  $w_{n+i}^*$  is the only odd degree generator in the correct degree. The only option is the expected result. Again, the pairing gives us a  $d_{\rho_n(i+1/2)}$  in degrees corresponding to  $z_{n+i+1}^*$  and  $w_{n+i+1/2}^*$ . There are no other options, so  $d_{\rho_n(i+1/2)}$  is as advertised.

Having computed these two must-have differentials, it gives us the description of  $E_{\rho_n(i+1/2)+1}$  of Theorem 8.1(ii).

We are not finished. We *must show that there are no extraneous  $d^r$ 's*. We assume, by induction, that we have  $E_{\rho_n(i-1/2)+1}$ . We have just found two necessary differentials between here and  $E_{\rho_n(i+1/2)+1}$ , our  $d^{\rho_n(i)}$  and  $d^{\rho_n(i+1/2)}$ . All we need to do now is show there are no other differentials,  $d^r$ , with  $\rho_n(i - 1/2) < r \leq \rho_n(i + 1/2)$ . If we take out the elements required for the two established differentials, what we have left is

$$(8.5) \quad P(y_{n,i+1}) \otimes E(w_{n,i+1}) \otimes W_{n,i+1} \otimes L_n \otimes TZ_{n,i+1}$$

for  $W_{n,i+1}$ ,  $L_n$  and  $TZ_{n,i+1}$  as in (2.6).

There are no differentials on the  $z$ 's because they are permanent cycles. This leaves only the  $w$ 's and  $y_{n,i+1}$ . The lowest odd degree element is  $w_{n,i+1}$  and we know that a differential on  $y_{n,i+1}$  would be  $\rho_n(i + 1) > \rho_n(i + 1/2)$ , so we cannot have a differential on  $y_{n,i+1}$ .

All that is left is to show there is no  $r$  in this range with  $d^r(w_{n,j+i+1}) \neq 0$ , for  $0 \leq j \leq n$ . Let  $r$  be the smallest  $r$  in the range  $\rho_n(i - 1/2) < r \leq \rho_n(i + 1/2)$  with  $d^r(w_{n,j+i+1}) = v_n^r \beta \neq 0$ , for the  $w_{n,j+i+1}$  of smallest degree. We know from Theorem 1.2, *The Pairing*, that there is a  $\beta'$ , with  $|\beta'| = |\beta|$ , in the homology Adams spectral sequence with  $d_r(\beta') = v_n^r \alpha' \neq 0$ . If  $\beta'$  is decomposable, then there must be an element,  $\beta''$ , with lower degree than  $\beta'$  with  $d_r(\beta'') = v_n^r \alpha'' \neq 0$ . For example, if  $\beta' = ab$ , then  $d_r(\beta') = d_r(a)b \pm ad_r(b)$  and either  $d_r(a)$  or  $d_r(b)$  is non-zero. In either case, we get our  $\beta''$  with degree less than  $|\beta'|$ . Again, by *The Pairing*, there is an  $\alpha$  in the cohomology Adams spectral sequence with  $|\alpha| = |\alpha''| < |w_{n,j+i+1}|$  with  $d^r(\alpha) \neq 0$ . This contradicts our choice of  $w_{n,j+i+1}$ . We conclude that if there

is such an  $r$ ,  $\beta'$  is indecomposable. Theorem 1.2, *The Pairing*, is pretty vague about what the corresponding elements are. All it really gives us are degrees.

Since we started with the odd degree  $w_{n,j+i+1}$ , we are looking for an even degree target element. However, we know where all the even degree indecomposables are in the homology spectral sequence, dual to (8.5). These elements are the  $y_s^*$ ,  $s > i$ , and the  $\gamma_{p^k}(z_{n+i+s+2}^*)$ , with  $k < n$ . We have similar looking elements in the (8.5), namely,  $y_{n,s}$ ,  $s > i$ , and  $z_{n+i+s+2}^{p^k}$ ,  $s \geq 0$ ,  $k < n$ . They are not known to be "dual" in any sense, but they are in the right degrees. All we will use about these cohomology elements is their degree. If we can show that there are no differentials that hit elements in these degrees, we are done.

We have three main ways to show a differential cannot exist. (1) We can use the *Divisibility Criterion*, (2) we can show that a prospective  $d^r$  has  $r > \rho_n(i + 1/2)$ , or (3) we can show that  $r \leq \rho_n(i - 1/2)$ .

First we have to check to see if there is some  $s$  with  $d^r(w_{n,j+i+1}) = v_n^r y_{n,s}$ , again, we repeat, only using the degree of  $y_{n,s}$ . For this, we must have

$$|w_{n,j+i+1}| + 1 + 2r(p^n - 1) = |y_{n,s}|$$

but we can replace the first term using (6.7)

$$|y_{n,j+i+1}| + 1 + 2\rho_n(j + i + 1)(p^n - 1) + 1 + 2r(p^n - 1) = |y_{n,s}|$$

so

$$|y_{n,s}| - |y_{n,j+i+1}| - 2 = 2p^s - 2p^{j+i+1} - 2$$

is both positive and divisible by  $2(p^n - 1)$ . Let  $s'$  and  $(j + i + 1)'$  be  $s$  and  $(j + i + 1)$  mod  $n$ . Then, mod  $2(p^n - 1)$ , this is  $2p^{s'} - 2p^{(j+i+1)'} - 2$ . This cannot be zero mod  $2(p^n - 1)$  so the *Divisibility Criterion* tells us we cannot have this differential.

The elements  $z_{n+i+s+2}^{p^k}$  below are in degrees that correspond to the degrees of the remaining even degree generators in the homology version. We have to show, using only their degrees, that there is **no** differential

$$(8.6) \quad d^r(w_{n,j+i+1}) = v_n^r z_{n+i+s+2}^{p^k} \quad \text{with} \\ 0 \leq j < n, 0 \leq s, 0 \leq k < n, 0 \leq i, \text{ and } \rho_n(i - 1/2) < r \leq \rho_n(i + 1/2).$$

We don't have to worry about the  $w$  with  $j = n$  because it was created by  $d^{\rho_n(i+1/2)}$ . (This is a degree reason.)

We will assume that  $i > 0$ . Our proof works for  $i = 0$  but there are parts that become degenerate and even easier.

We replace  $|w_{n,j+i+1}|$  with  $|y_{n,j+i+1}| + 1 + 2\rho_n(j + i + 1)(p^n - 1)$ . so we have

$$|y_{n,j+i+1}| + 1 + 2\rho_n(j + i + 1)(p^n - 1) + 1 + 2r(p^n - 1) = |z_{n+i+s+2}^{p^k}|.$$

Turning this into numbers and rearranging,

$$\begin{aligned} 2(p^n - 1)(\rho_n(j + i + 1) + r) &= |z_{n+i+s+2}^{p^k}| - |y_{n,j+i+1}| - 2 \\ &= p^k 2(p^{n+i+s+2} + 1) - 2p^{j+i+1} - 2 \\ &= 2(p^{n+i+s+k+2} + p^k) - 2p^{j+i+1} - 2. \end{aligned}$$

It is  $r$  we are interested in, so rewrite

$$(8.7) \quad 2r(p^n - 1) = 2p^{n+i+s+k+2} + 2p^k - 2p^{j+i+1} - 2 - 2\rho_n(j + i + 1)(p^n - 1).$$

For starters, we consider the case when  $s = j - k$ . Both  $j$  and  $k$  are less than  $n$ , so our  $s$  is small. Of course this only makes sense when  $k \leq j$ . We want to show that  $\rho_n(i + 1/2) < r$ . This eliminates all  $p^{k'}$  powers where  $k' > k$  as well. Also, when  $k = 0$ , it eliminates  $s = j$ , and consequently, all  $s \geq j$  and all  $k$ .

We prefer to work with  $\rho_n(-)$  for whole values of the variable, so we replace  $\rho_n(i + 1/2) = p^{i+1} - \rho_n(i)$  (from (6.5)) for integers  $i$  and  $s = j - k$  in (8.7). We want

$$(p^{i+1} - \rho_n(i))2(p^n - 1) < 2p^{n+j+i+2} + 2p^k - 2p^{j+i+1} - 2 - 2\rho_n(j + i + 1)(p^n - 1),$$

which rearranges to

$$p^{i+1}2(p^n - 1) + 2\rho_n(j + i + 1)(p^n - 1) + 2p^{j+i+1} + 2 < 2p^{n+j+i+2} + 2p^k + 2\rho_n(i)(p^n - 1).$$

Because  $\rho_n(j + i + 1) \leq p^{j+i+1}$  (from (6.5)) it is enough to show

$$p^{i+1}2(p^n - 1) + 2p^{j+i+1}(p^n - 1) + 2p^{j+i+1} + 2 < 2p^{n+j+i+2} + 2p^k + \rho_n(i)2(p^n - 1).$$

Expand and rearrange:

$$2p^{n+j+i+1} + 2p^{n+i+1} + 2 < 2p^{n+j+i+2} + 2p^k + \rho_n(i)2(p^n - 1) + 2p^{i+1}$$

This is pretty clearly always true, so we are done with this.

This concludes our argument when  $s \geq j$  for all  $k$ . All that is left is if  $s < j$ . For this, we set  $k = j - s$ . Now, again, we let  $s = j - k$ . The previous computation took care of the  $p^k$ -th power of the  $z$ , and so also when  $k' > k$ . We still have to deal with powers when  $k' < k$ . This is done in two steps. First we show that using the power  $p^{k-1}$ , the differential cannot exist because of the *Divisibility Criterion*, not because it is too short or too long. After that, we show that if we take the power  $p^{k-2}$ , still setting  $s = j - k$  as above, the differential is too short. This solves the problem when the powers have  $k' \leq k - 2$ , because they will also be too short. That will take care of all the cases listed in (8.6).

Using (8.7) but with  $p^{k-1}$  instead of  $p^k$ ,

$$\begin{aligned} 2r(p^n - 1) &= 2p^{n+j+i+1} + 2p^{k-1} - 2p^{j+i+1} - 2 - 2\rho_n(j + i + 1)(p^n - 1) \\ &= 2(p^n - 1)(p^{j+i+1} - \rho_n(j + i + 1)) + 2p^{k-1} - 2. \end{aligned}$$

Because  $k - 1 < n - 1$ , we cannot solve for  $r$  unless  $k = 1$ .

The  $k = 1$  case corresponds to  $d^r(w_{n,j+i+1}) = v_n^r z_{n+j+i+1}$ . For this, we need another lemma.

**Lemma 8.8.** *If  $d^r(w_{n,j+i+1}) = v_n^r z_{n+j+i+1}$ ,  $0 < i$ ,  $0 \leq j < n$ , then  $r \leq \rho_n(i - 1/2)$ .*

*Proof.* We have already found that  $r = p^{j+i+1} - \rho_n(j + i + 1)$ , so we need

$$p^{j+i+1} \leq \rho_n(j + i + 1) + \rho_n(i - 1/2).$$

We did the  $j = 0$  case in the proof of Lemma 8.4 and the proof here is similar. This is easy to see with small  $i$  and we use Lemma 6.4 and induction to see the rest. The inductive step is to show

$$p^{n+j+i+2} \leq \rho_n(n + j + i + 2) + \rho_n(n + i + 1/2).$$

The right hand side is

$$p^{j+i+2}(p^n - 1) + 1 + \rho_n(j + i + 1) + p^{n+i}(p - 1) - 1 + \rho_n(i - 1/2)$$

which is greater than or equal to

$$p^{j+i+2}(p^n - 1) + p^{n+i}(p - 1) + p^{j+i+1}.$$

Multiplying out and rearranging, this is greater than or equal to  $p^{n+j+i+2}$  if

$$p^{n+i+1} + p^{j+i+1} \geq p^{j+i+2} + p^{n+i}.$$

Recall that  $j < n$ . This is clearly true when  $j < n - 1$ . When  $j = n - 1$  we get an equality.  $\square$

So, we have eliminated the  $s = j - k$  and  $p^{k-1}$  case. All we did here was show this could not exist. We did not show that it was too short or too long if it did exist. So, this doesn't solve the problem for  $p^{k'}$  powers with  $k' < k - 1$ .

We show that the differential would be too short if we use  $p^{k-2}$ , i.e., that  $r \leq \rho_n(i - 1/2)$ . If we do that, then all the lower powers work as well.

For this we go back to (8.7) but now with  $p^{k-2}$  instead of  $p^k$ .

$$\begin{aligned} 2r(p^n - 1) &= 2p^{n+j+i} + 2p^{k-2} - 2p^{j+i+1} - 2 - 2\rho_n(j + i + 1)(p^n - 1) \\ &= 2p^{k-2} - 2 - 2\rho_n(j + i + 1)(p^n - 1). \end{aligned}$$

We want this to be less than  $2\rho_n(i - 1/2)(p^n - 1)$ , or

$$2p^{k-2} - 2 - 2\rho_n(j + i + 1)(p^n - 1) \leq 2\rho_n(i - 1/2)(p^n - 1),$$

which we can rewrite as

$$2p^{k-2} \leq 2\rho_n(i - 1/2)(p^n - 1) + 2 + 2\rho_n(j + i + 1)(p^n - 1).$$

Recall that  $n > k$ , so if  $\rho_n(j + i + 1) > 0$ , this is true.

This concludes all of the cases we needed to check. There are no more differentials than those already produced.

Of course if there are no more differentials in the Adams spectral sequence for  $k(n)^*(K_2)$ , then *The Pairing* says there are no more for  $k(n)_*(K_2)$ .  $\square$

## 9. FROM COHOMOLOGY TO HOMOLOGY

We now turn from  $k(n)^*(K_2)$  to  $k(n)_*(K_2)$ . We have already given  $H_*K_2$  in (7.4). We need the  $Q_n$  homology of  $H_*K_2$ , but it is just dual to the  $Q_n$  homology of  $H^*K_2$  described in Section 4. It gives us the  $E_2$  term of the Adams spectral sequence for  $k(n)_*(K_2)$ , which we spell out in Theorem 9.4(i).

We give the  $E_2$  term of the Adams spectral sequence for  $k(n)_*(K_2)$  and describe all the differentials in Theorem 9.4, and give the final result as a  $k(n)_*$ -module in Theorem 9.5. The proofs are dual to the proofs for  $k(n)^*(K_2)$ .

Using the notation of (3.6), we have elements

$$(9.1) \quad \begin{aligned} v_n &\in G_2^{2(p^n-1),1} = E_2^{1,2(p^n-1)+1}, \\ y_{n,j}^* &\in G_2^{2p^j,0} = E_2^{0,2p^j}, \\ w_{n,i}^* &\in G_2^{2p^{n+i}+1,0} = E_2^{0,2p^{n+i}+1} \\ \text{and } z_{j,s}^* &= \gamma_{p^s}(z_j^*) \in G_2^{2p^s(p^j+1),0} = E_2^{0,2p^s(p^j+1)}. \end{aligned}$$

The dual analog of (2.6) is

$$(9.2) \quad \begin{aligned} EE_n^* &:= E(u_s^* : 0 \leq s < n) \otimes E(u_{2n+s}^* : s > 0), \\ W_{n,i}^* &:= E(w_{n,i+s}^* : 1 \leq s \leq n), \\ L_n^* &:= \bigotimes_{0 < s < n} \Gamma_{p^{n-s}}(z_s^*), \\ TZ_{n,i}^* &:= \Gamma_{p^n}(z_{n+s}^* : s > i), \quad \text{and} \\ PZ_n &:= \Gamma(\gamma_{e_n(s)}(z_s^*) : s > 0) \quad \text{with } e_n(s) \text{ as in (2.6)}. \end{aligned}$$

We also need dual analogs of (2.9) and (2.11).

$$\begin{aligned}
D_1^* &:= \Gamma_p(y_{n,0}^*) \otimes E(w_{n,0}^*) \otimes EE_n^* \otimes PZ_n^*, \\
S_{n,0}^* &:= D_1^*, \\
M_{n,i}^* &:= \Gamma(y_{n,i+1}^*) \otimes W_{n,i}^* \otimes L_n^* \otimes TZ_{n,i}^* \quad \text{for } i \geq 0, \\
M_{n,i+1/2}^* &:= \Gamma(y_{n,i+1}^*) \otimes W_{n,i}^* \otimes L_n^* \otimes TZ_{n,i+1}^* \quad \text{for } i \geq 0, \\
S_{n,i}^* &:= T_{\rho_n(i)}(v_n) \otimes \overline{y_{n,i}^* \Gamma_{p-1}(y_{n,i}^*)}, \\
&= T_{\rho_n(i)}(v_n) \otimes \{\gamma_s(y_{n,i}^*) : 1 \leq s \leq p-1\} \quad \text{for } i > 0, \\
\text{and } S_{n,i+1/2}^* &:= T_{\rho_n(i+1/2)}(v_n) \otimes \overline{z_{n+i+1}^* \Gamma_{p-1}(z_{n+i+1}^*)} \\
&= T_{\rho_n(i+1/2)}(v_n) \otimes \{\gamma_s(z_{n+i+1}^*) : 1 \leq s \leq p^n-1\} \quad \text{for } i \geq 0.
\end{aligned}
\tag{9.3}$$

Using this notation, we can now state the dual of Theorems 4.3 and 8.1.

**Theorem 9.4.** **The Adams  $E_2$ -term, differentials and intermediate terms for  $k(n)_*(K_2)$ .**

(i) *The  $E_2$  term of the odd primary Adams spectral sequence for  $k(n)_*(K_2)$  is*

$$P(v_n) \otimes L_n^* \otimes \Gamma(y_{n,1}^*) \otimes E(y_{n,1/2}^*) \otimes W_{n,0}^* \otimes TZ_{n,0}^*$$

*plus a computable family of filtration-0  $\mathbf{Z}/p$ 's annihilated by  $v_n$  coming from the dual of the  $E(Q_n)$ -free part of  $H_*K_2$ , specified in Theorem 2.12.*

(ii) *In the odd primary Adams spectral sequence for  $k(n)_*(K_2)$ , the differentials  $d_r$  for  $r \geq 1$  are*

$$\begin{aligned}
d_1(u_n^*) &= v_n y_{n,0}^*, \\
d_1(\gamma_{p^s}(z_{n-s}^*)) &= v_n u_s^* && \text{for } 0 \leq s < n, \\
d_1(\gamma_{p^n}(z_{s-n}^*)) &= v_n u_s^* && \text{for } s > 2n, \\
d_{\rho_n(i+1/2)}(z_{n+i+1}^*) &= v_n^{\rho_n(i+1/2)} y_{n,i+1/2}^* && \text{for } i \geq 0, \\
\text{and } d_{\rho_n(i)}(w_{n,i}^*) &= v_n^{\rho_n(i)} y_{n,i}^* && \text{for } i > 0.
\end{aligned}$$

(iii) *For each  $\ell \geq 0$ ,*

$$\begin{aligned}
E_{1+\rho_n(\ell/2)} &= E_{\rho_n((\ell+1)/2)} \\
&= \bigoplus_{0 \leq k \leq \ell} (S_{n,k/2}^* \otimes M_{n,k/2}^*)
\end{aligned}$$



$$\oplus \left( k(n)^* \otimes \left\{ \begin{array}{l} E(y_{n,(\ell+1)/2}^*) \otimes \Gamma_{p^n}(z_{n+1+\ell/2}^*) \\ \text{for } \ell \text{ even} \\ \Gamma_p(y_{n,(\ell+1)/2}^*) \otimes E(w_{n,(\ell+1)/2}^*) \\ \text{for } \ell \text{ odd} \end{array} \right\} \otimes M_{n,(\ell+1)/2}^* \right)$$

for  $M_{n,\ell/2}^*$  and  $S_{n,\ell/2}^*$  as in (9.3).

The dual of Theorem 2.12 is

**Theorem 9.5.** For an odd prime  $p$ ,  $k(n)_*(K_2)$  has the following three summands as a  $k(n)_*$ -module:

- (i) The  $k(n)_*$  free summand,  $k(n)^* \otimes L_n^*$ , for  $L_n$  as in (9.2).
- (ii) The higher torsion summand,

$$\bigoplus_{\ell > 0} (M_{n,\ell/2}^* \otimes S_{n,\ell/2}^*),$$

for  $M_{n,\ell/2}^*$  and  $S_{n,\ell/2}^*$  as in (9.3).

- (iii) The elementary torsion summand,  $S_{n,0}^* \otimes M_{n,0}^*$  as in (9.3).

These follow from Theorems 8.1 and 2.12 using the duality of Theorem 1.2.

## 10. MODIFICATIONS FOR $p = 2$

All we do in this section is to lay out the results for  $k(n)_*(K_2)$  for  $p = 2$ . We skip the homology version and proofs. We do this with a twinge of guilt. The very first case done was the  $p = 2, n = 1$  case, and there, the generally useful *Divisibility Criterion* is worthless. Consequently, there are lots of little *ad hoc* arguments that must be done in that case.

For  $p = 2$ ,  $H^*K_2 = P(\iota_2) \otimes_{i \geq 0} P(u_i)$ , with  $u_i = Q_i \iota_2$ . We let  $u_i^2 = z_{i+1} = Q_{i+1} Q_0 \iota_2$  ( $z_0 = 0$ ). In an attempt to be as similar as possible with notation, we have

$$|u_i| = 2 \times 2^i + 1$$

$$\text{and } |z_i| = 2(2^i + 1) \quad \text{for } i > 0$$

$$|Q_n| = 2 \times 2^n - 1, y_{n,j} = \iota_2^{2^j} \quad \text{for } j \geq 0 \text{ in degree } 2 \times 2^j$$

with

$$Q_n u_i = \begin{cases} (u_{n-i-1})^{2^{i+1}} = z_{n-i}^{2^i} & \text{for } 0 \leq i < n \\ 0 & \text{for } n \leq i \leq 2n \\ (u_{i-n-1})^{2^{n+1}} = z_{i-n}^{2^n} & \text{for } i > 2n \end{cases}$$

$$Q_n y_{n,0} = u_n$$

The formulas used for odd primes mostly work here

$$w_{n,i} := \begin{cases} u_n & \text{for } i = 0 \\ u_{n+i} + u_{n-i} z_i^{2^n - 2^{n-i}} & \text{for } 0 < i \leq n \\ u_{2n+1} + y_{n,1/2} z_{n+1}^{2^n - 1} & \text{for } i = n + 1 \\ y_{n,i-n-1/2} z_i^{2^n - 1} & \text{for } i > n + 1 \end{cases}$$

$$y_{n,j+1/2} := y_{n,j} w_{n,j}.$$

The only difference between the above and (2.2) is the definition of  $w_{n,n+1}$ , which here includes  $u_{2n+1}$ . For odd primes we have

$$d^1 u_{n+1} = v_n z_{n+1}^{p^n}, \quad \text{and} \quad d^{p-1} y_{n,1/2} = v_n^{p-1} z_{n+1}$$

making the heuristic expression

$$w_{n,n+1} := y_{n,1/2} z_{n+1}^{p^n - 1} - v_n^{p-2} u_{n+1}$$

a cycle. We do not see the second term for  $p > 2$  because it has higher filtration, but for  $p = 2$  both terms have filtration 0.

We can compute the  $\mathbb{Q}_n$ -homology of  $H^* K_2$  with a diagram like that of (4.1).

**Theorem 10.1.** *We have elements  $v_n \in G_{-2(2^n-1),1}^2$ ,  $y_{n,1} \in G_{4,0}^2$ ,  $w_{n,i} \in G_{2^{n+i+1}+1,0}^2$ , and  $z_j \in G_{2^{j+1}+2,0}^2$ . The  $E_2$  term of the  $p = 2$  Adams spectral sequence for  $k(n)^*(K_2)$  is*

$$\begin{aligned} & P(v_n) \otimes \bigotimes_{0 < i < n} T_{2^i} z_{n-i} \otimes P(y_{n,1}) \\ & \otimes T_{2^{n+1}}(w_{n,i+1} : 0 \leq i \leq n) \otimes T_{2^n}(z_{2n+1+i+1} : i > 0) \end{aligned}$$

plus a computable family of filtration-0  $\mathbb{Z}/2$ 's annihilated by  $v_n$  coming from the  $E(\mathbb{Q}_n)$ -free part of  $H^* K_2$ .

For convenience we reset  $z_{n+i+1} = w_{n,i}^2$  for  $0 < i \leq n + 1$ .

**Proposition 10.2.** *For  $p = 2$ , the differentials in the  $p = 2$  Adams spectral sequence for  $k(n)^*(K_2)$  are:*

(i) *For  $0 < j \leq n + 1$ ,  $\rho_n(j) = 2^j = \rho_n(j + 1/2)$ . Although  $y_{n,j+1/2} = y_{n,j} w_{n,j}$ , for  $j \leq n + 1$ , this is not a generator.*

$$d^{2^j}(y_{n,j}) = v_n^{2^j} w_{n,j}$$

$$\text{and} \quad d^{2^j}(y_{n,j} w_{n,j}) = v_n^{2^j} w_{n,j}^2 = v_n^{2^j} z_{n+j+1}.$$

(ii) *For  $j > n + 1$ ,*

$$d^{\rho_n(j)}(y_{n,j}) = v_n^{\rho_n(j)} w_{n,j}$$

$$\text{and} \quad d^{\rho_n(j+1/2)}(y_{n,j+1/2}) = v_n^{\rho_n(j+1/2)} z_{n+j+1}.$$

(iii) For  $0 < j \leq n+1$ ,  $\rho_n(j) = 2^j = \rho_n(j+1/2)$ . Ignoring the permanent free terms and the previously created  $v_n$ -torsion,

$$E_{2^{j+1}} = k(n)^* \otimes P(y_{n,j+1}) \otimes \bigotimes_{j \leq i \leq n} T_{2^{n+1}}(w_{n,i+1}) \\ \otimes \bigotimes_{0 \leq i < j} E(w_{2n+2+i}) \otimes \bigotimes_{0 < s} T_{2^n}(z_{2n+2+s}).$$

(iv) For  $n+1 < j$ ,

$$E_{\rho_n(j)+1} = k(n)^* \otimes P(y_{n,j+1}) \otimes E(y_{n,j+1/2}) \\ \otimes \bigotimes_{0 < i \leq n} E(w_{n,j+i}) \otimes \bigotimes_{0 \leq s} T_{2^n}(z_{n+j+s+1})$$

We could rewrite  $T_{2^{n+1}}(w_{n,i+1})$  as  $E(w_{n,i+1}) \otimes T_{2^n}(z_{n+i+2})$  for  $0 \leq i \leq n$ . If we did that, we could write proposition 10.2 without the exceptional cases. Since our interest is in the  $k(n)^*$ -module structure and not so much in the multiplicative structure, we do this for our final result.

**Theorem 10.3.** *The 2-primary  $k(n)^*(K_2)$  as a  $k(n)^*$ -module is the sum of the following three summands:*

$$P(v_n) \otimes \bigotimes_{0 < i < n} T_{2^{n-i}}(z_{i+1}) \\ \bigoplus_{j > 0} \left( T_{\rho_n(j)}(v_n) \otimes P(y_{n,j+1}) \otimes \bar{E}(w_{n,j}) \otimes \bigotimes_{0 < i \leq n} E(w_{n,j+i}) \otimes \bigotimes_{s \geq 0} T_{2^n}(z_{n+j+s+1}) \right) \\ \bigoplus_{j > 0} \left( T_{\rho_n(j+1/2)}(v_n) \otimes P(y_{n,j+1}) \otimes \bigotimes_{0 < i \leq n} E(w_{n,j+i}) \right. \\ \left. \otimes \bar{TP}_{2^n}(z_{n+j+1}) \otimes \bigotimes_{s > 0} T_{2^n}(z_{n+j+s+1}) \right)$$

plus a computable family of  $\mathbf{Z}/2$ 's annihilated by  $v_n$  coming from the  $E(Q_n)$ -free part of  $H^*K_2$ .

## REFERENCES

- [AH68] D. W. Anderson and L. Hodgkin. The  $K$ -theory of Eilenberg-Mac Lane complexes. *Topology*, 7(3):317–330, 1968.
- [D] D. M. Davis. The connective  $KO$ -theory of the Eilenberg-MacLane space  $K(\mathbf{Z}/2, 2)$ . in preparation.

- [DW24a] D.M. Davis and W.S. Wilson. The connective KO-theory of the Eilenberg-MacLane space  $K(\mathbb{Z}_2, 2)$ , I: The  $E_2$  page. 2024. Submitted.
- [DW24b] D. M. Davis and W. S. Wilson, The connective  $K$ -theory of the Eilenberg-MacLane space  $K(\mathbb{Z}/p, 2)$ . *Glasgow Math Jour* 66:188–220, 2024.
- [Laz01] A. Lazarev. Homotopy theory of  $A_\infty$  ring spectra and applications to  $MU$ -modules. *K-Theory*, 24(3):243–281, 2001.
- [May69] J. Peter May. Matric Massey products. *J. Algebra*, 12:533–568, 1969.
- [MRS01] Mark E. Mahowald, Douglas C. Ravenel, and Paul L. Shick. The triple loop space approach to the telescope conjecture. In *Homotopy methods in algebraic topology (Boulder, CO, 1999)*, volume 271 of *Contemp. Math.*, pages 217–284. Amer. Math. Soc., Providence, RI, 2001.
- [Pst23] Piotr Pstragowski. Synthetic spectra and the cellular motivic category. *Invent. Math.*, 232(2):553–681, 2023.
- [Rav86] Douglas C. Ravenel. *Complex cobordism and stable homotopy groups of spheres*, volume 121 of *Pure and Applied Mathematics*. Academic Press Inc., Orlando, FL, 1986. Second edition available online at author’s home page.
- [Rav04] Douglas C. Ravenel. *Complex Cobordism and Stable Homotopy Groups of Spheres, Second Edition*. American Mathematical Society, Providence, 2004. Available online at the author’s home page.
- [Rob87] Alan Robinson. Spectra of derived module homomorphisms. *Math. Proc. Cambridge Philos. Soc.*, 101(2):249–257, 1987.
- [Rob89] Alan Robinson. Obstruction theory and the strict associativity of Morava  $K$ -theories. In *Advances in homotopy theory (Cortona, 1988)*, volume 139 of *London Math. Soc. Lecture Note Ser.*, pages 143–152. Cambridge Univ. Press, Cambridge, 1989.
- [RW80] Douglas C. Ravenel and W. Stephen Wilson. The Morava  $K$ -theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *Amer. J. Math.*, 102(4):691–748, 1980.
- [RW77] Douglas C. Ravenel and W. Stephen Wilson. The Hopf ring for complex cobordism. *J. Pure Appl. Algebra*, 9(3):241–280, 1976/77.
- [RWY98] Douglas C. Ravenel, W. Stephen Wilson, and Nobuaki Yagita. Brown-Peterson cohomology from Morava  $K$ -theory. *K-Theory*, 15(2):147–199, 1998.
- [Tam97] Hirotaka Tamanoi. The image of the BP Thom map for Eilenberg-Mac Lane spaces. *Trans. Amer. Math. Soc.*, 349(3):1209–1237, 1997.
- [Tam99] Hirotaka Tamanoi.  $\mathcal{Q}$ -subalgebras, Milnor basis, and cohomology of Eilenberg-Mac Lane spaces. *J. Pure Appl. Algebra*, 137(2):153–198, 1999.

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