

# THE HOPF RING FOR $P(n)$

DOUGLAS C. RAVENEL AND W. STEPHEN WILSON

ABSTRACT. We show that  $E_*(\underline{P(n)})_*$ , the  $E$ -homology of the  $\Omega$ -spectrum for  $P(n)$ , is an  $E_*$  free Hopf ring for  $E$  a complex oriented theory with  $I_n$  sent to 0. This covers the cases  $P(q)_*(\underline{P(n)})_*$  and  $K(q)_*(\underline{P(n)})_*$ ,  $q \geq n$ . The generators of the Hopf ring are those necessary for the stable result. The motivation for this paper is to show that  $P(n)$  satisfies all of the conditions for the machinery of unstable cohomology operations set up in [BJW95]. This can then be used to produce splittings analogous to those for  $BP$  done in [Wil75].

Mathematics Subject Classification Number: Primary: 55N22, Secondary 55N20.

## 1. INTRODUCTION

The spectrum  $P(n)$  for  $n > 0$  is the  $BP$ -module spectrum obtained by killing the ideal

$$I_n = (p, v_1, v_2, \dots, v_{n-1}) \subset \pi_*(BP)$$

via the Sullivan-Baas construction, [Baa73], [BM71], and [JW75]. For odd primes it is a nice multiplicative spectrum by [Mor79], [SY76], and [Wür77]. It comes equipped with a stable cofibration

$$(1.1) \quad \Sigma^{2(p^n-1)}P(n) \longrightarrow P(n) \longrightarrow P(n+1)$$

which gives the following short exact sequence in homotopy

$$0 \longrightarrow P(n)_* \xrightarrow{v_n} P(n)_* \longrightarrow P(n+1)_* \longrightarrow 0.$$

The  $i^{\text{th}}$  space in the  $\Omega$ -spectrum for  $P(n)$  will be denoted by  $\underline{P(n)}_i$ . Because  $P(n)$  is a ring spectrum there are maps

$$\underline{P(n)}_i \times \underline{P(n)}_j \longrightarrow \underline{P(n)}_{i+j}$$

corresponding to cup product, in addition to the loop space product

$$\underline{P(n)}_i \times \underline{P(n)}_i \longrightarrow \underline{P(n)}_i.$$

These induce pairings

$$\circ : E_*(\underline{P(n)}_i) \otimes E_*(\underline{P(n)}_j) \rightarrow E_*(\underline{P(n)}_{i+j})$$

and

$$* : E_*(\underline{P(n)}_i) \otimes E_*(\underline{P(n)}_i) \rightarrow E_*(\underline{P(n)}_i).$$

for a generalized homology theory  $E_*$ . If  $E_*$  has a Künneth isomorphism for these spaces, e.g. if they are  $E_*$  free, then these pairings satisfy certain identities, making  $E_*(\underline{P(n)})_*$  into a *Hopf ring* [RW77], i.e., a ring object in the category of coalgebras. *The object of this paper is to describe this structure explicitly for suitable theories  $E_*(-)$ , namely, when  $E$  is a complex orientable spectrum with  $I_n = 0$ .*

---

Partially supported by the National Science Foundation.

In the next section we will define some special elements

$$\begin{aligned}
e &\in P(n)_1(\underline{P(n)}_1), \\
a_{(i)} &\in P(n)_{2p^i}(\underline{P(n)}_1) \quad \text{for } 0 \leq i < n, \\
[v_i] &\in P(n)_0(\underline{P(n)}_{-2(p^i-1)}) \quad \text{for } i \geq n, \quad \text{and} \\
b_{(i)} &\in P(n)_{2p^i}(\underline{P(n)}_2) \quad \text{for } i \geq 0,
\end{aligned}$$

which have already been defined in previous papers.

Let

$$e^\varepsilon a^I [v^K] b^J = e^\varepsilon \circ a_{(0)}^{\circ i_0} \circ \cdots \circ a_{(n-1)}^{\circ i_{n-1}} \circ [v_n^{k_n} v_{n+1}^{k_{n+1}} \cdots] \circ b_{(0)}^{\circ j_0} \circ b_{(1)}^{\circ j_1} \cdots$$

where  $\varepsilon = 0$  or  $1$ ,  $i_q = 0$  or  $1$ ,  $k_q \geq 0$ , and  $j_q \geq 0$  ( $K$  and  $J$  finite).

**Definition 1.2.** We say  $e^\varepsilon a^I [v^K] b^J$  is  $n$ -allowable if

$$J = p^n \Delta_{d_n} + p^{n+1} \Delta_{d_{n+1}} + \cdots + p^m \Delta_{d_m} + J'$$

where  $\Delta_d$  has a 1 in the  $d^{\text{th}}$  place and zeros elsewhere,  $d_n \leq d_{n+1} \leq \cdots \leq d_m$  and  $J'$  is non-negative implies  $k_m = 0$ . In other words,

$$[v_m] \circ b^{p^n \Delta_{d_n} + p^{n+1} \Delta_{d_{n+1}} + \cdots + p^m \Delta_{d_m}}$$

does not divide  $e^\varepsilon a^I [v^K] b^J$  when  $d_n \leq d_{n+1} \leq \cdots \leq d_m$ . We will denote the set of such  $(K, J)$  by  $\mathcal{A}_n$ .

We say  $e^\varepsilon a^I [v^K] b^J$  is  $n$ -plus allowable if  $e^\varepsilon a^I [v^K] b^{J+\Delta_0}$  is  $n$ -allowable. We will denote the set of such  $(K, J)$  by  $\mathcal{A}_n^+$ . Note that  $\mathcal{A}_n^+ \subset \mathcal{A}_n$ .

Note that when restricted to elements where  $k_0$  equals 0, 0-plus allowable as defined above coincides with allowable as defined in [RW77], (and is the same as 1-allowable with  $i_0 = 0$ ). For  $n = \infty$ , the allowability condition is vacuous.

Let  $TP_k(x) = P(x)/(x^{p^k})$ ; we say that such an  $x$  has height  $p^k$ . Let  $E$  be a  $BP$ -module spectrum. Then we have a map  $BP \xrightarrow{\mu} E$ . For  $p$  an odd prime,  $P(n)$  is the universal multiplicative  $BP$ -module spectrum with  $\mu_*(I_n) = 0$ . This fact follows from the work of Würigler in [Wür77, p. 477, 6.8]. Thus if  $E$  has  $\mu_*(I_n) = 0$ , then the map  $\mu$  factors through  $P(n)$ .

**Theorem 1.3.** *Let  $E$  be a multiplicative  $BP$ -module spectrum with  $\mu_*(I_n) = 0$ ,  $n > 0$ . Let  $p$  be an odd prime. As  $E_*$ -algebras:*

$$E_*(\underline{P(n)}_*) \simeq \left\{ \begin{array}{l} \bigotimes_{(K,J) \in \mathcal{A}_n^+} E(ea^I[v^K]b^J) \\ \bigotimes \\ \bigotimes_{\substack{i_0=0 \\ (K,J) \in \mathcal{A}_n}} TP_{\rho(I)}(a^I[v^K]b^J) \\ \bigotimes \\ \bigotimes_{\substack{I \neq M, i_0=1 \\ (K,J) \in \mathcal{A}_n^+}} TP_{\rho(I)}(a^I[v^K]b^J) \\ \bigotimes \\ \bigotimes_{(K,J) \in \mathcal{A}_n^+} P(a^M[v^K]b^J) \end{array} \right.$$

where  $\rho(I)$  is the smallest  $t$  with  $i_{n-t} = 0$ , and  $M$  is the exponent sequence consisting entirely of ones, i.e.,  $a^M = a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)}$ .

This calculation includes the calculation of  $K(q)_*(\underline{P(n)}_*)$  for  $q \geq n$ . For  $q < n$ , this calculation was carried out in [HRW, Theorem 1.5]. The results have very little in common.

If we restrict to elements with  $k_0 = 0$  then the theorem is also true as stated for  $n = 0$ ,  $P(0)$  being  $BP$  by definition. In that case there are no  $a_{(i)}$ s, so we always have  $I = M = 0$ , which means the truncated polynomials factors are trivial. As remarked above, 0-plus allowability (for elements with  $k_0 = 0$ ) is the same as allowability as defined in [RW77], so Theorem 1.3 coincides with the main theorem of [RW77]. Note that  $P(1)$  is just  $BP \bmod p$ .

For  $n = \infty$  the theorem gives the usual Hopf ring description of the homology of mod  $p$  Eilenberg–Mac Lane spaces. As remarked above, the allowability condition is vacuous in this case. There is no polynomial factor since  $M$  is infinite,  $K = 0$ , and  $\rho(I) = 1$  for all  $I$ , so the truncated polynomial algebras all have height  $p$ . See [Wil82, §8] for more details.

The theorem will be proved by studying the bar spectral sequence (see §3) for mod  $p$  homology going from  $H_*(\underline{P(n)}_i)$  to  $H_*(\underline{P(n)}_{i+1})$ . We then show that the Atiyah–Hirzebruch spectral sequence collapses for appropriate  $E_*(-)$ . For  $n = 0$  and  $n = \infty$  the bar spectral sequence always collapses, but not for  $0 < n < \infty$ . There are no multiplicative extensions for  $n = \infty$ . The extensions differ significantly between the  $n = 0$  case, studied in [RW77] and the  $n > 0$  cases studied here which more resemble those in [Wil84]. We will give examples to illustrate in the next section.

The main theorem above contains a description of the generators. No description of Hopf algebras is complete without understanding the primitives. Although Theorem 1.3 contains all the information about the coproduct in principle, we can be more explicit. Furthermore, we need a more explicit description during our proof.

Both Theorem 1.3 and our results about primitives will be proven inductively simultaneously by degree.

**Theorem 1.4.** *Let  $E$  be as in Theorem 1.3. The primitives in  $E_*(\underline{P(n)})$  have the following description:*

- (a) *A basis for the primitive elements is given by all  $e^\varepsilon a^I [v^K] b^J$  such that*
  - (i) *if  $\varepsilon = 1$  then  $(K, J) \in \mathcal{A}_n^+$ ,*
  - (ii) *if  $\varepsilon = 0$  then  $i_0 + j_0 > 0$  and  $(K, J) \in \mathcal{A}_n$ .*
- (b) *A basis for the primitive elements of height  $p$  is given by all of the above primitives  $a^I [v^K] b^J$  such that  $i_{n-1} = 0$ .*
- (c) *In the mod  $p$  homology,  $H_*(\underline{P(n)})$ , the iterated  $p$ -th powers of the generators in Theorem 1.3 are all primitives. They are those primitives with  $\varepsilon = 0$  and  $i_0 = 1$  with  $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$  modulo the vector space generated by  $a^I [v^K] b^J$  with  $i_0 = 1$  and  $(K, J) \in \mathcal{A}_n^+$  (which are primitive generators).*

Part (a) of the theorem still holds for the  $n = 0$  case. Part (c) is about the nontrivial  $p^{\text{th}}$  powers. Some generators are not primitive and so it is interesting to note that all of their  $p^{\text{th}}$  powers are primitive.

*From now on we assume that  $n > 0$ .*

Our proof of 1.3 will follow the lines of several previous papers; in particular, [RW77], [RW80], and [Wil84]. As in [Wil84], it is enough to prove the result for ordinary mod  $p$  homology and then show the Atiyah-Hirzebruch spectral sequence collapses for  $E$ . Consequently, we will focus most of our attention on ordinary homology. In fact, our theorem and proof lies somewhere between the work in [Wil84] and [RW77]. In turn, the work of [Wil84] lies somewhere between [RW77] and [RW80].

Our proof follows the philosophy of the second author that one can compute the homology of spaces in an  $\Omega$ -spectrum if one knows the stable homotopy and the stable homology. The homotopy gives the zero dimensional homology of all the spaces and if one computes by induction on degree using the bar spectral sequence any false computation of a differential or extension should lead to a contradiction with the stable homology. This works in many cases, including this one. Getting a nice Hopf ring description is an entirely different matter. It seems to depend on having the stable elements appear at the earliest possible stage unstably.

There are a number of Hopf rings like this that have been computed. Some are “good” and some are “bad”. Examples of good ones are  $E_*(\underline{BP}_*)$ ,  $E$  a complex orientable theory, [RW77];  $E_*(\underline{K(n)})$ ,  $E$  a complex orientable theory with  $I_n = 0$ , [Wil84];  $H_*(K(\mathbf{Z}/(p), *))$ , [Wil82, §8];  $K(n)_*(-)$  for Eilenberg–Mac Lane spaces, [RW80];  $K(n)_*(\underline{k(n)})$ , [Kra90];  $H_*(\underline{KO})$ , [Str92]; and more recently the breakthrough description of  $H_*(QS^0, \mathbf{Z}/(2))$  in [Tur], and its sequel for  $H_*(QS^*, \mathbf{Z}/(2))$  in [ETW]. All of these examples can have their Hopf rings described with just a few generators and relations. There are other similar calculations where the Hopf rings are not so nice, for example for  $bo$ ,  $bu$ ,  $BP\langle n \rangle$ , and  $k(n)$ . The standard mod  $p$  homology of these does not work out so well as a Hopf ring. Despite that obstacle, the results for  $bu$  and  $k(n)$  have been given very nice descriptions in [Har91].

By [Wür77] and [Yag77] we know that  $P(n)_*(P(n))$  is free over  $P(n)_*$ . This result is all that is necessary to show that  $P(n)$  satisfies all of the machinery for stable operations as in [Boa95]. Our results show:

**Corollary 1.5.** *Both  $P(n)_*(\underline{P(n)}_*)$  and the module of indecomposables,  $QP(n)_*(\underline{P(n)}_*)$ , are free over  $P(n)_*$ .*

These two new conditions are enough to make all of the machinery of unstable operations in [BJW95] work, which was the motivation for this paper. In [BJW95], this machinery is used to prove the second author's splitting theorem for the spaces in the  $\Omega$ -spectrum for  $BP$ , [Wil75]. The second author had conjectured a similar splitting for  $P(n)$  and this paper together with [BJW95] allows that splitting to be carried through; see [BW]. The lowest cases involve Morava  $K$ -theories; e.g.  $\underline{P(n)}_{2p^n} \simeq \underline{k(n)}_{2p^n} \times Y$ .

We would like to thank the referee for his or her careful reading, particularly for pointing out a serious error in our notation. The referee also suggested we do a global version of Theorem 1.3 but this has already been written into [BW].

The  $p = 2$  case deserves some discussion. As the theorem is stated it is true for  $p = 2$  as well. In particular it is true for mod 2 homology. The problem is that all the spectra  $E$  that we care about, such as  $K(n)$  or  $P(n)$ , are not commutative ring spectra. Even in those cases the result is true as modules if not as algebras because we get the general  $E$  from the collapsing of the Atiyah-Hirzebruch spectral sequence. As in [Wil84], no problems are caused because of the lack of commutativity of  $P(n)$  in  $\underline{P(n)}_*$ . We make other comments about  $p = 2$  in the next section.

## 2. BASIC PROPERTIES

We have a long proposition analogous to [Wil84, Proposition 1.1].

**Proposition 2.1.** *Let  $E$  be as in Theorem 1.3. Let  $p$  be an odd prime. We have elements*

$$\begin{aligned} e &\in E_1(\underline{P(n)}_1), \\ a_i &\in E_{2i}(\underline{P(n)}_1) \quad \text{for } 0 \leq i < p^n, \\ [v_i] &\in E_0(\underline{P(n)}_{-2(p^i-1)}) \quad \text{for } i \geq n, \quad \text{and} \\ b_i &\in E_{2i}(\underline{P(n)}_2) \quad \text{for } i \geq 0 \end{aligned}$$

such that (letting  $b_{(i)} = b_{p^i}$  and  $a_{(i)} = a_{p^i}$ ):

- (a) *They are natural with respect to  $E$ .*
- (b) *The homomorphism  $x \mapsto e \circ x$  is the homology suspension map.*
- (c) *The coproduct is given by  $a_i \rightarrow \sum a_{i-j} \otimes a_j$  and  $b_i \rightarrow \sum b_{i-j} \otimes b_j$ .*
- (d) *They are all permanent cycles in the Atiyah-Hirzebruch spectral sequence for  $P(n)_*(\underline{P(n)}_*)$ .*
- (e)  $e \circ e = -b_1$ .
- (f)  $a_{(i)} \circ a_{(j)} = -a_{(j)} \circ a_{(i)}$ .
- (g)  $b_{(i)}^{*p} = 0$ .
- (h)  $a_{(i)}^{*p} = 0$  for  $i < n - 1$ .
- (i)  $a_{(n-1)}^{*p} = \mu_*(v_n)a_{(0)} - a_{(0)} \circ [v_n] \circ b_{(0)}^{\circ p^n - 1}$ .
- (j)  $\mu_*(v_n)e = e \circ [v_n] \circ b_{(0)}^{\circ p^n - 1}$ .

*Proof.* The proof of this proposition is identical to that of [Wil84, Proposition 1.1]. In fact, that paper uses  $P(n)$  in the proof. For some of the results one only

needs to notice that  $P(n)$  and  $k(n)$  are the same up to degree  $2(p^{n+1} - 1)$  and that the proposition takes place in low degrees. A sign has crept into (e) because of a correction from [BJW95].  $\square$

We will recall briefly the construction of these elements.

- Let  $L$  denote the  $(2p^n - 2)$ -skeleton of  $K(\mathbf{Z}/(p), 1)$ . Then there is a unique lifting

$$\begin{array}{ccc} & \underline{P(n)}_1 & \\ & \nearrow & \downarrow \\ L & \longrightarrow & K(\mathbf{Z}/(p), 1) \end{array}$$

and hence a map  $E_*(L) \rightarrow E_*(\underline{P(n)}_1)$ . The elements  $e$  and  $a_i$  are the images of the standard elements in  $E_*(L)$ .

- $[v_k]$  is the Hurewicz image of

$$v_k \in \pi_{2p^k-2}(P(n)) = \pi_0(\underline{P(0)}_{-2p^k+2}).$$

- There is a canonical generator of  $P(n)^2(\mathbf{C}P^\infty)$ , which corresponds to a map  $\mathbf{C}P^\infty \rightarrow \underline{P(n)}_2$ . The elements  $b_i$  are the images under this map of the standard classes in  $E_*(\mathbf{C}P^\infty)$ .

**The case  $p = 2$ .** The comments of [Wil84] are relevant here.  $e$  should be incorporated into the  $a$ 's for coproduct purposes and  $a_{(i)} \circ a_{(i)}$  is not zero. One must fiddle with the proof a little, but not much.

The proof of our theorem will rely on being able to identify elements in the bar spectral sequence, compute differentials and solve multiplicative extension problems, all using Hopf ring techniques. The  $n = 0$  case of [RW77] has no differentials but does have extension problems. For the bar spectral sequence going from  $H_*(\underline{P(0)}_i)$  to  $H_*(\underline{P(0)}_{i+1})$  there are extensions when  $i$  is odd. We will illustrate this phenomenon with an example for  $i = 1$ . For  $k \geq 0$  let

$$g_k = e \circ [v_1]^k \circ b_{(0)}^{op-1} \circ \cdots \circ b_{(k-1)}^{op-1} \in H_*(\underline{P(0)}_1).$$

Then  $H_*(\underline{P(0)}_1)$  has the exterior algebra

$$E(g_0, g_1, \cdots)$$

as a factor, and the corresponding Tor group contains the divided power algebra

$$\Gamma(\sigma g_0, \sigma g_1, \cdots)$$

as a factor; see 3.2. Now for  $k > 0$  we have (in ordinary mod  $p$  homology)

$$\begin{aligned}
\sigma g_k &= e \circ e \circ [v_1^k] \circ \overline{b_{(0)}^{\circ p-1}} \circ \cdots \circ b_{(k-1)}^{\circ p-1} && \text{by 2.1(b)} \\
&= -b_{(0)} \circ [v_1^k] \circ b_{(0)}^{\circ p-1} \circ \cdots \circ b_{(k-1)}^{\circ p-1} && \text{by 2.1(e)} \\
&= -([v_1] \circ b_{(0)}^{\circ p}) \circ [v_1^{k-1}] \circ b_{(1)}^{\circ p-1} \circ \cdots \circ b_{(k-1)}^{\circ p-1} \\
&= b_{(0)}^{*p} \circ [v_1^{k-1}] \circ b_{(1)}^{\circ p-1} \circ \cdots \circ b_{(k-1)}^{\circ p-1} && \text{using 2.6} \\
&= ([v_1] \circ b_{(0)}^{\circ p})^{*p} \circ [v_1^{k-2}] \circ b_{(2)}^{\circ p-1} \circ \cdots \circ b_{(k-1)}^{\circ p-1} \\
&\quad \text{by the Hopf ring distributive law [RW77, 1.12(c)(vi)]} \\
&= -b_{(0)}^{*p^2} \circ [v_1^{k-2}] \circ b_{(2)}^{\circ p-1} \circ \cdots \circ b_{(k-1)}^{\circ p-1} && \text{using 2.6} \\
&\vdots \\
&= \pm b_{(0)}^{*p^k}.
\end{aligned}$$

It follows that the divided power factor above in the  $E_\infty$ -term corresponds to a polynomial factor

$$P(b_{(0)}, b_{(1)}, \dots)$$

in  $H_*(\underline{P(0)}_2)$  and also in  $E_*(\underline{P(0)}_2)$ .

For  $\underline{P(n)}$  with  $n > 0$  this type of extension never occurs because of 2.1(g). The type of extension we have is more interesting because they frequently lead to truncated polynomial algebras and are implied by the relations 2.1(h) and (i). Since we work in ordinary mod  $p$  homology  $\mu(v_n) = 0$ . For  $n = 1$ , 2.1(i) says

$$\begin{aligned}
a_{(0)}^{*p} &= -a_{(0)} \circ [v_1] \circ b_{(0)}^{\circ p-1} \\
\text{so } a_{(0)}^{*p^2} &= (-a_{(0)} \circ [v_1] \circ b_{(0)}^{\circ p-1})^{*p} \\
&= -(a_{(0)})^{*p} \circ [v_1] \circ b_{(1)}^{\circ p-1} \\
&\quad \text{by the Hopf ring distributive law} \\
&= a_{(0)} \circ [v_1^2] \circ b_{(0)}^{\circ p-1} \circ b_{(1)}^{\circ p-1} \\
a_{(0)}^{*p^3} &= -a_{(0)} \circ [v_1^3] \circ b_{(0)}^{\circ p-1} \circ b_{(1)}^{\circ p-1} \circ b_{(2)}^{\circ p-1} \\
&\vdots
\end{aligned}$$

so  $a_{(0)} \circ x$  for suitable  $x$  could be a polynomial generator.

For  $n = 2$  we have

$$\begin{aligned}
a_{(0)}^{*p} &= 0 \\
a_{(1)}^{*p} &= -a_{(0)} \circ [v_2] \circ b_{(0)}^{\circ p^2 - 1} \\
\text{so } a_{(1)}^{*p^2} &= (-a_{(0)} \circ [v_2] \circ b_{(0)}^{\circ p^2 - 1})^{*p} \\
&= -(a_{(0)})^{*p} \circ [v_2] \circ b_{(1)}^{\circ p^2 - 1} \\
&= 0 \\
\text{and } (a_{(0)} \circ a_{(1)})^{*p} &= a_{(1)} \circ (a_{(0)})^{*p} \\
&= -a_{(1)} \circ a_{(0)} \circ [v_2] \circ b_{(0)}^{\circ p^2 - 1} \\
&= a_{(0)} \circ a_{(1)} \circ [v_2] \circ b_{(0)}^{\circ p^2 - 1} \\
\text{so } (a_{(0)} \circ a_{(1)})^{*p^2} &= a_{(0)} \circ a_{(1)} \circ [v_2^2] \circ b_{(0)}^{\circ p^2 - 1} \circ b_{(1)}^{\circ p^2 - 1} \\
&\vdots
\end{aligned}$$

and for larger  $n$  the same thing happens with powers of  $a^M = a_{(0)} \circ a_{(1)} \circ \cdots \circ a_{(n-1)}$ . *This accounts for the polynomial factor in Theorem 1.3.*

Also observe that for  $n = 3$ , 2.1(i) gives

$$\begin{aligned}
(a_{(0)} \circ a_{(1)})^{*p} &= a_{(1)} \circ (a_{(0)}^{*p}) \\
&= 0, \\
(a_{(0)} \circ a_{(2)})^{*p} &= a_{(1)} \circ (a_{(2)})^{*p} \\
&= -a_{(1)} \circ (a_{(0)} \circ [v_3] \circ b_{(0)}^{\circ p^3 - 1}) \\
&= a_{(0)} \circ a_{(1)} \circ [v_3] \circ b_{(0)}^{\circ p^3 - 1} \\
\text{so } (a_{(0)} \circ a_{(2)})^{*p^2} &= 0, \\
\text{and } (a_{(1)} \circ a_{(2)})^{*p} &= a_{(2)} \circ (a_{(1)})^{*p} \\
&= -a_{(2)} \circ (a_{(0)} \circ [v_3] \circ b_{(0)}^{\circ p^3 - 1}) \\
&= a_{(0)} \circ a_{(2)} \circ [v_3] \circ b_{(0)}^{\circ p^3 - 1} \\
\text{so } (a_{(1)} \circ a_{(2)})^{*p^3} &= 0.
\end{aligned}$$

*This (and similar computations for larger  $n$ ) accounts for the truncations in Theorem 1.3.*

For the spectral sequence computations of the next section, we will need the following results about  $p^{\text{th}}$  powers.

Define shift operators  $s$  on  $I$  (if  $i_{n-1} = 0$ ) and  $J$  by

$$\begin{aligned}
(2.2) \quad a^{s(I)} &= a_{(1)}^{\circ i_0} \circ a_{(2)}^{\circ i_1} \circ \cdots \circ a_{(n-1)}^{\circ i_{n-2}} \quad \text{and} \\
b^{s(J)} &= b_{(1)}^{\circ j_0} \circ b_{(2)}^{\circ j_1} \circ \cdots .
\end{aligned}$$

**Lemma 2.3.** *In the mod  $p$  homology of  $\underline{P(n)}_*$ , let  $(K, J)$  be in  $\mathcal{A}_n$ ,*

*(a) then*

$$(a^I [v^K] b^J)^{*p} = 0 \quad \text{if} \quad i_{n-1} = 0,$$



(b) if  $i_{n-1} = 1$  let  $I = I'' + \Delta_{n-1}$ . We have

$$(a^I [v^K] b^J)^{*p} = \pm a^{\Delta_0 + s(I'')} [v^{\Delta_n + K}] b^{(p^n - 1)\Delta_0 + s(J)},$$

and

(c) If  $I = M$  as in 1.3,

$$(a^M [v^K] b^J)^{*p} = \pm a^M [v^{\Delta_n + K}] b^{(p^n - 1)\Delta_0 + s(J)}.$$

This is not much information right now because the potentially nonzero  $p^{\text{th}}$  powers are not usually in allowable form. The introduction of the  $[v_n]$  together with the possibility that some  $j_k \geq p^n$  messes this up. However, this does give us some useful information. If  $I$  is not  $M$  (as in 1.3) then the iterated  $p^{\text{th}}$  power must eventually be zero but in the case (c) we could have a polynomial generator. Even though we do not know, at this stage, if these  $p^{\text{th}}$  powers are really nonzero, we know something about their properties if they are.

**Corollary 2.4.** *Each nonzero  $p^{\text{th}}$  power in Lemma 2.3 is divisible by  $a_{(0)}$  and is primitive.*

*Proof of Lemma 2.3.* Assume first that  $I$  is empty and  $J = \Delta_m + J'$  where  $j_m$  is the first non-zero index in  $J$ . Then using the Hopf ring distributive law and 2.1(g) we have

$$\begin{aligned} ([v^K] b^J)^{*p} &= (b_{(m)}^{*p}) \circ [v^K] b^{s(J')} \\ &= 0. \end{aligned}$$

Similarly if  $I = \Delta_m + I'$  (where  $m$  is the first nonzero index in  $I$ ) and  $i_{n-1} = 0$ , we have, using 2.1(h)

$$\begin{aligned} (a^I [v^K] b^J)^{*p} &= (a_{(m)}^{*p}) \circ a^{s(I')} [v^K] b^{s(J)} \\ &= 0. \end{aligned}$$

However, when  $i_{n-1} = 1$ ,  $a^{s(I')}$  is not defined, so we must proceed differently. We write  $a^I = a^{I''} \circ a_{(n-1)}$  and use 2.1(i) and get

$$\begin{aligned} (a^I [v^K] b^J)^{*p} &= \pm (a_{(n-1)}^{*p}) \circ a^{s(I'')} [v^K] b^{s(J)} \\ &= \pm (a_{(0)} \circ [v_n] \circ b_{(0)}^{\circ p^n - 1}) \circ a^{s(I'')} [v^K] b^{s(J)} \\ &= \pm a^{\Delta_0 + s(I'')} [v^{\Delta_n + K}] b^{(p^n - 1)\Delta_0 + s(J)}. \end{aligned}$$

Part (c) follows from part (b).  $\square$

*Proof of Corollary.* Observation tells us that our elements are divisible by  $a_{(0)}$ . Since  $a_{(0)}$  is a primitive and we know that circle product with a primitive is a primitive, all of our elements are primitive.  $\square$

We now have to vary a little from [Wil84]. When we do so, we need only go back to [RW77] to find what we need. Let us work, as always when we use standard homology, in mod  $p$  homology. Let  $Q$  stand for the indecomposables and  $[I_\infty] = ([v_n], [v_{n+1}], \dots)$ . Then we have:

**Theorem 2.5.** *In  $QH_*(P(n)_2)/[I_\infty]^{\circ 2} \circ QH_*(P(n)_*)$  we have:*

$$\sum_{i=n}^k [v_i] \circ b_{(k-i)}^{\circ p^i} = 0.$$

*Proof.* The proof is the same as the proof of [RW77, Theorem 3.14, page 259].  $\square$

This follows from the *main relation*, [RW77, Theorem 3.8], which covers our case and can be rewritten as:

**Theorem 2.6.** *Let  $b(t) = \sum b_i t^i$ . Then, in  $P(n)_* \underline{P(n)}_* [[s]]$ ,*

$$b \left( \sum_{j \geq n}^F v_j s^{p^j} \right) = *_{j \geq n}^{[F]} [v_j] \circ b(s)^{\circ p^j}.$$

We need another piece of [RW77] which was not needed in [Wil84]. Namely, we need a theorem that allows us to reduce non-basis elements to basis elements. The relation we use to do this is Theorem 2.5, but it is not an easy one to apply.

**Theorem 2.7.** *In  $QH_*(P(n)_*)$ , any  $e^\varepsilon a^I [v^K] b^J$  can be written in terms of  $n$ -allowable elements.*

*Proof.* The construction and proof of an algorithm for the reduction of non-allowable elements is done on pages 273–275 of [RW77]. The proof applies with only notational modification to the case of  $n$ -allowable when  $I = 0$ . We can then circle multiply by  $a^I$  to get our result.  $\square$

Theorems 1.3 and 1.4 are stated for rather general  $E$  as described in Theorem 1.3. We do all of our calculations in mod  $p$  homology and so we must lift our results to  $E$ .

*Proof of Theorems 1.3 and 1.4 for general  $E$  from the theorems for mod  $p$  homology.* It is enough, for Theorem 1.3, to show that the Atiyah-Hirzebruch spectral sequence collapses. The Atiyah-Hirzebruch spectral sequence respects the two products,  $\circ$  and  $*$ , and all elements in  $P(n)_*(P(n)_*)$  are constructed using these two products from the basic elements of Proposition 2.1. Since the basic elements are all permanent cycles by Proposition 2.1 (d), every element is a permanent cycle and the spectral sequence collapses. The elements of Theorem 1.4 (a) are all primitive and no more can be created. Part (b) also still holds. The only concern is the possibility that for  $P(n)$  the truncated polynomial generators do not truncate at the same place because of a shift in filtrations. However, Proposition 2.1 (h) and (i) tells us that the height of an element is determined strictly by the pattern of its  $a$ 's. The results for general  $E$  follow by naturality from those for  $P(n)$ .  $\square$

### 3. THE SPECTRAL SEQUENCE

All that remains is to prove Theorems 1.3 and 1.4 for mod  $p$  homology. We prove our two theorems simultaneously by induction on degree in the bar spectral sequence. Recall that for a loop space  $X$  with classifying space  $BX$  the bar spectral sequence converges to  $H_*(BX)$ , and its  $E^2$ -term is

$$\mathrm{Tor}_{*,*}^{H_*(X)}(\mathbf{Z}/(p), \mathbf{Z}/(p));$$

we will abbreviate this group by  $\mathrm{Tor}_{*,*}^{H_*(X)}$  or  $H_{*,*}(H_*(X))$ . When  $BX$  is also a loop space, we have a spectral sequence of Hopf algebras. (See the discussion in [HRW, §2].)

In this section we collect all of the facts that describe the complete behavior of this spectral sequence in the case  $X = \underline{P(n)}_i$ , including its differentials and solutions to the algebra extension problems. Our proof is similar to that of [Wil84]

throughout except that we need a little more information from time to time. We could have mimiced the proof completely but instead we make some improvements.

We will prove Theorems 1.3 and 1.4 by induction on degree using the bar spectral sequence with Hopf rings, [TW80]. We let  $\underline{P(n)}'_j$  be the zero component of  $\underline{P(n)}_j$ . Then  $\underline{P(n)}_j \simeq P(n)^j \times \underline{P(n)}'_j$ , where  $P(n)^j$  denotes the group  $\pi_{-j}(P(n))$ . We assume the calculations in Theorems 1.3, 1.4 and the results stated below (specifically Lemma 3.6 and Theorem 3.7) are correct for  $H_i(\underline{P(n)}_*)$  with  $i < k$ . (The value of  $H_0(\underline{P(n)}_*)$  is obvious, given our knowledge of  $\pi_*(P(n))$ .) The bar spectral sequence determines  $H_i(\underline{P(n)}'_{*+1})$  for  $i \leq k$  and so we get  $H_i(\underline{P(n)}_*)$  for  $i \leq k$ . This induction is always with us, although frequently only implicitly. Let  $\sigma$  denote the suspension in the spectral sequence and  $\phi$  the transpotent. Most of our notation goes back to [RW80].

For future use let

$$(3.1) \quad m(J) = \min\{k: j_k \neq 0\},$$

and define  $m(I)$  similarly.

When we are working in  $H_0(\underline{P(n)}_*)$  we really need  $[v_i] - [0_{-2(p^i-1)}]$  to get it into the augmentation ideal. See [RW77] for further details. In positive degrees this does not affect anything.

The following standard result enables us to compute all of the relevant Tor groups.

**Proposition 3.2.** *The group  $\text{Tor}^K(\mathbf{Z}/(p), \mathbf{Z}/(p)) \equiv \text{Tor}^K$  for a graded  $\mathbf{Z}/(p)$ -algebra  $K$  has the following properties:*

- (i) *It commutes with tensor products, i.e.,*

$$\text{Tor}^{K_1 \otimes K_2} = \text{Tor}^{K_1} \otimes \text{Tor}^{K_2}.$$

- (ii) *For  $K = E(x)$  (an exterior algebra on an odd dimensional generator  $x$ ),*

$$\text{Tor}^{E(x)} = \Gamma(\sigma x),$$

*a divided power algebra on the suspension of  $x$ , with  $\gamma_i(\sigma x) \in \text{Tor}_{i, i|x|}^{E(x)}$  represented in the algebraic bar construction by  $\otimes^i x$ . As an algebra,*

$$\Gamma(\sigma x) = TP_1(\sigma x, \gamma_p(\sigma x), \gamma_{p^2}(\sigma x), \dots)$$

*where  $\gamma_0(\sigma x) = \sigma x$ .*

- (iii) *For  $K = P(x)$  (a polynomial algebra on an even dimensional generator  $x$ ),*

$$\text{Tor}^{P(x)} = E(\sigma x),$$

*an exterior algebra on the suspension of  $x$ ,  $\sigma x \in \text{Tor}_{1, |x|}^{P(x)}$ .*

- (iv) *For  $K = TP_k(x)$  (a truncated polynomial algebra of height  $p^k$  on an even dimensional generator  $x$ ),*

$$\text{Tor}^K = E(\sigma x) \otimes \Gamma(\phi(x^{p^{k-1}})),$$

*where  $\phi(y)$  is the transpotent of  $y$ , with  $\gamma_i(\phi(y)) \in \text{Tor}_{2i, pi|y|}^K$ . A representative cycle for  $\phi(y)$  is  $y^{p-1} \otimes y$  and for  $\gamma_i(\phi(y))$ ,  $\otimes^i (y^{p-1} \otimes y)$ .*

The Tor groups corresponding to the homology given in Theorem 1.3 are given below in Lemma 3.6.

In our proof we will need to be able to identify elements in the spectral sequence. We usually cannot do this precisely but must introduce some filtrations. In particular, we need to introduce an entirely new filtration. To do that we need to review, following [TW80], how the Hopf ring pairing fits into the bar spectral sequence. We recall how this pairing is constructed. The bar spectral sequence converging to  $H_*(\underline{P}(n)_{q+1})$  is based on the bar filtration of the space

$$\underline{P}(n)_{q+1} = B\underline{P}(n)_q = \bigcup_i B_i \underline{P}(n)_q$$

where the  $B$  in the middle here is for the classifying space. In [TW80] it was shown that the circle product respects this filtration, i.e., the map

$$\underline{P}(n)_{q+1} \wedge \underline{P}(n)_r \xrightarrow{\circ} \underline{P}(n)_{q+1+r}$$

induces maps

$$B_i \underline{P}(n)_q \wedge \underline{P}(n)_r \xrightarrow{\circ} B_i \underline{P}(n)_{q+r}$$

for each  $i$ . In the bar filtration we have cofibre sequences

$$B_{i-1} \underline{P}(n)_q \longrightarrow B_i \underline{P}(n)_q \longrightarrow \Sigma^i \underline{P}(n)_q^{(i)}$$

where this last space is the  $i^{\text{th}}$  suspension of the  $i$ -fold smash product of  $\underline{P}(n)_q$ . It follows that the pairing induces maps

$$(3.3) \quad \Sigma^i \underline{P}(n)_q^{(i)} \wedge \underline{P}(n)_r \longrightarrow \Sigma^i \underline{P}(n)_{q+r}^{(i)}.$$

This map is the usual circle product on each of the  $i$  factors.

Now recall the Verschiebung map  $V$ , defined on any cocommutative coalgebra as the dual of the  $p^{\text{th}}$  power map. Since our bar spectral sequence is one of bicommutative Hopf algebras, it has a Verschiebung map

$$(3.4) \quad E_{ps,pt}^r \xrightarrow{V} E_{s,t}^r$$

which, if one ignores the grading, is a Hopf algebra map. In addition, it respects the circle product pairing, i.e.,

$$V(x \circ y) = (Vx) \circ (Vy).$$

The Verschiebung is a standard tool; we will also need the following variant of it. There is an *internal Verschiebung*

$$H_{s,pt}(H_*(X)) \xrightarrow{V_{int}} H_{s,t}(H_*(X))$$

defined on the algebraic bar construction, i.e.  $E_{*,*}^1$ , by

$$V_{int}(x_1 \otimes x_2 \otimes \cdots \otimes x_s) = Vx_1 \otimes Vx_2 \otimes \cdots \otimes Vx_s.$$

Because  $V(x * y) = V(x) * V(y)$  we see that  $V_{int}$  commutes with  $d^1$  and is defined on  $E_{*,*}^2$ , our Tor for the spectral sequence. Reviewing the definition of the Hopf algebra structure on Tor we see that  $V_{int}$  is, ignoring the gradings, a Hopf algebra map. From (3.3) we can deduce that  $V_{int}$  satisfies the identity

$$(3.5) \quad V_{int}((x_1 \otimes x_2 \otimes \cdots \otimes x_s) \circ y) = (V_{int}(x_1 \otimes x_2 \otimes \cdots \otimes x_s)) \circ Vy.$$

Now we are ready to identify the Tor group for the algebra of Theorem 1.3.

**Lemma 3.6.** *Let  $p$  be any prime. In the bar spectral sequence,*

$$E_{*,*}^r(H_*(\underline{P(n)}_*)) \Rightarrow H_*(\underline{P(n)}'_{*+1}),$$

The Tor group for the algebra of Theorem 1.3 is as follows.

$$\begin{aligned} E_{*,*}^2(H_*(\underline{P(n)}_*)) &\simeq \text{Tor}_{*,*}^{H_*(\underline{P(n)}_*)}(\mathbf{Z}/(p), \mathbf{Z}/(p)) \equiv H_{*,*}(H_*(\underline{P(n)}_*)) \simeq \\ &\bigotimes_{(K,J) \in \mathcal{A}_n^+} \Gamma(\sigma e \circ a^I [v^K] b^J) \\ &\quad \otimes \\ &\bigotimes_{\substack{i_0=0 \\ (K,J) \in \mathcal{A}_n}} \Gamma(\sigma a^I [v^K] b^J) \\ &\quad \otimes \\ &\bigotimes_{\substack{i_0=1 \\ (K,J) \in \mathcal{A}_n^+}} \Gamma(\sigma a^I [v^K] b^J) \\ &\quad \otimes \\ &\bigotimes_{\substack{(K,J) \in \mathcal{A}_n \\ i_{n-1}=0}} \Gamma(\phi(a^I [v^K] b^J)). \end{aligned}$$

*Proof.* The Tor group for each factor in 1.3 can be computed using Proposition 3.2. The result is shown in the following table.

Factor	Tor group
$\bigotimes_{(K,J) \in \mathcal{A}_n^+} E(ea^I [v^K] b^J)$	$\bigotimes_{(K,J) \in \mathcal{A}_n^+} \Gamma(\sigma ea^I [v^K] b^J)$
$\bigotimes_{\substack{i_0=0 \\ (K,J) \in \mathcal{A}_n}} TP_{\rho(I)}(a^I [v^K] b^J)$ $\otimes$ $\bigotimes_{\substack{I \neq M, i_0=1 \\ (K,J) \in \mathcal{A}_n^+}} TP_{\rho(I)}(a^I [v^K] b^J)$	$\bigotimes_{\substack{i_0=0 \\ (K,J) \in \mathcal{A}_n}} E(\sigma a^I [v^K] b^J)$ $\otimes$ $\bigotimes_{\substack{I \neq M, i_0=1 \\ (K,J) \in \mathcal{A}_n^+}} E(\sigma a^I [v^K] b^J)$ $\otimes$ $\bigotimes_{\substack{(K,J) \in \mathcal{A}_n \\ i_{n-1}=0}} \Gamma(\phi(a^I [v^K] b^J))$
$\bigotimes_{(K,J) \in \mathcal{A}_n^+} P(a^M [v^K] b^J)$	$\bigotimes_{(K,J) \in \mathcal{A}_n^+} E(\sigma a^M [v^K] b^J)$

The only difficulty here is identifying the transpotent elements. From Lemma 2.3 we get the  $i_{n-1}$  condition. From Theorem 1.4 we can find all of the elements

of height  $p$  in the truncated polynomial factors of Theorem 1.3 except the non-primitive generators which can then be read off directly from Theorem 1.3.

We can consolidate the third and fifth factors on the right.  $\square$

We will need to identify the generators,  $\gamma_{p^i}$ , in our Tor using the Hopf ring pairing. We can only do this modulo filtrations which is adequate for our needs. The first filtration is given by the Verschiebung where we have that  $V^i(\gamma_{p^i}(\sigma x)) = \gamma_0(\sigma x)$  and we see that  $V^i$  kills all other elements in this filtration except those of this form. In this case,  $V^i$  kills all decomposables so when we work modulo the kernel of  $V^i$  we are just working modulo decomposables. However, when we work with  $\gamma_{p^i}(\phi(y))$ ,  $V^i$  does not kill elements of the form  $\gamma_{p^i}(\sigma x_1) * \gamma_{p^i}(\sigma x_2)$  but these terms will not show up and we can still work modulo decomposables.

Our second filtration comes from  $V_{int}$  and it is important for us to see how it acts on our Tor groups.  $V$  kills any primitive and anything with an  $e$ ,  $a_{(0)}$  or  $b_{(0)}$  is a primitive so for starters,  $V_{int}$  kills the first and third factors of Lemma 3.6. When using this filtration the exterior elements from the second factor will never enter in. All we need to be concerned with then is what happens with the fourth factor. If  $i_0$  or  $j_0$  is greater than zero then our factor goes to zero. If both are zero then we can define the inverse shift operator,  $s^{-1}$  on  $I$  and  $J$ . Since  $V(a_{(i+1)}) = a_{(i)}$  and  $V(b_{(i+1)}) = b_{(i)}$ , we have that  $V_{int}$  induces an isomorphism:

$$\Gamma(\phi(a^I[v^K]b^J)) \longrightarrow \Gamma(\phi(a^{s^{-1}(I)}[v^K]b^{s^{-1}(J)})).$$

We will be able to identify elements modulo the kernel of the  $V_{int}^i$  which we now understand.

We can now describe the behavior of the entire bar spectral sequence.

**Theorem 3.7.** *Let  $p$  be any prime.*

- (a) *In the Hopf ring pairing of the bar spectral sequence we have:  
For  $J \neq 0$  and  $k = m(J)$ , consider*

$$\circ : H_{*,*}(H_*(\underline{P(n)}_{*2})) \otimes H_*(\underline{P(n)}_2) \rightarrow H_{*,*}(H_*(\underline{P(n)}_*)).$$

- (i)  $\gamma_{p^i}(\sigma e a^I[v^K]b^{J-\Delta_k}) \circ b_{(k+i)} = \gamma_{p^i}(\sigma e a^I[v^K]b^J)$  modulo decomposables for  $(K, J) \in \mathcal{A}_n^+$ .  
(ii)  $\gamma_{p^i}(\phi(a^I[v^K]b^{J-\Delta_k})) \circ b_{(k+i+1)} = \gamma_{p^i}(\phi(a^I[v^K]b^J))$  modulo decomposables and the kernel of  $V_{int}^q$  where  $q = \min(m(I), m(J))$ , for  $(K, J) \in \mathcal{A}_n$  and  $i_{n-1} = 0$ .

- (b) *For  $J = 0$ ,  $I \neq 0$ , and  $k = m(I)$ , consider*

$$\circ : H_{*,*}(H_*(\underline{P(n)}_{*-1})) \otimes H_*(\underline{P(n)}_1) \rightarrow H_{*,*}(H_*(\underline{P(n)}_*)).$$

- (i)  $\gamma_{p^i}(\sigma e a^{I-\Delta_k}[v^K]) \circ a_{(k+i)} = (-1)^{l(I)-1} \gamma_{p^i}(\sigma e a^I[v^K])$  modulo decomposables,  $k+i < n$ .  
(ii)  $\gamma_{p^i}(\phi(a^{I-\Delta_k}[v^K])) \circ a_{(k+i+1)} = (-1)^{l(I)} \gamma_{p^i}(\phi(a^I[v^K]))$  modulo decomposables and the kernel of  $V_{int}^k$ , for  $k+i+1 < n$  and  $i_{n-1} = 0$ .  
(c) *Let  $q = \min\{j | i_{n-j} = 1\}$  if  $I \neq 0$ , and  $n+1$  if  $I = 0$ . The following differentials are nonzero:*  
(i)  $d^{p^q-1}$  on  $\gamma_{p^q}(\sigma e a^I[v^K]b^J)$ ,  $I \neq 0$  and  $(K, J) \in \mathcal{A}_n^+$ .  
(ii)  $d^{2p^q-1}$  on  $\gamma_{p^q-1}(\phi(a^I[v^K]b^J))$ ,  $(K, J) \in \mathcal{A}_n$  and  $i_{n-1} = 0$ .  
(d) *The differential targets of (c) are all linearly independent and, modulo the vector space generated by the  $\sigma a^I[v^K]b^J$  with  $i_0 = 0$  and  $(K, J) \in \mathcal{A}_n^+$ , a*

basis for the vector space they generate is given by all  $\sigma a^I[v^K]b^J$  with  $i_0 = 0$  such that  $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$ .

- (e) Let  $q = \min \{j | i_{n-j} = 1\}$ , up to sign in  $E_{*,*}^\infty$
- (i)  $\gamma_{p^i}(\sigma ea^I[v^K]b^J)$ ,  $(K, J) \in \mathcal{A}_n^+$ , represents  $a^{s^i(I)}[v^K]b^{s^i(J+\Delta_0)}$  modulo decomposables where if  $I \neq 0$  then  $i < q$ .
  - (ii)  $\gamma_{p^i}(\phi(a^I[v^K]b^J))$ , for  $(K, J) \in \mathcal{A}_n$ ,  $i_{n-1} = 0$ , represents the element  $a^{s^{i+1}(I)+\Delta_i}[v^K]b^{s^{i+1}(J)}$  modulo decomposables and the kernel of  $V_{int}^{\min(m(I), m(J))}$ , where  $i < q - 1$ . (If  $I = 0$  then  $q = n + 1$ .)
  - (iii)  $\sigma a^I[v^K]b^J$  represents  $ea^I[v^K]b^J$  when  $(K, J) \in \mathcal{A}_n^+$ .
- (f) As an algebra,  $E_{*,*}^\infty$  is

$$\bigotimes_{(K, J) \in \mathcal{A}_n^+} E(ea^I[v^K]b^J) \otimes \bigotimes_{(K, J) \in \mathcal{A}_n} TP_1(a^I[v^K]b^J)$$

*Proof.* (a). First we must note that both the  $\gamma_{p^i}$  elements exist in the spectral sequence and they do.

(a)(i). Next we work modulo decomposables so we can apply  $V^i$  to both sides and we need only show the  $i = 0$  case, or,

$$\sigma ea^I[v^K]b^{J-\Delta_k} \circ b_{(k)} = \sigma ea^I[v^K]b^J$$

for  $(K, J) \in \mathcal{A}_n^+$  which is obvious.

(a)(ii). First we apply  $V^i$  to see we need only show

$$\phi(a^I[v^K]b^{J-\Delta_k}) \circ b_{(k+1)} = \phi(a^I[v^K]b^J).$$

Next we want to apply  $V_{int}^q$ . If  $k = m(J) = q$  then we have

$$\begin{aligned} & \phi(V^q(a^I[v^K]b^{J-\Delta_k})) \circ b_{(1)} \\ &= (V^q(a^I[v^K]b^{J-\Delta_k})^{*p-1} \otimes V^q(a^I[v^K]b^{J-\Delta_k})) \circ b_{(1)} \\ &= \phi(V^q(a^I[v^K]b^J)). \end{aligned}$$

Since there are only  $p$  nontrivial terms in the iterated coproduct of  $b_{(1)}$  this is trivial. If  $k = m(J) > m(I) = q$  then we want to show

$$\begin{aligned} & \phi(V^q(a^I[v^K]b^{J-\Delta_k})) \circ b_{(1+k-q)} \\ &= (V^q(a^I[v^K]b^{J-\Delta_k})^{*p-1} \otimes V^q(a^I[v^K]b^{J-\Delta_k})) \circ b_{(1+k-q)} \\ &= \phi(V^q(a^I[v^K]b^J)). \end{aligned}$$

The argument is now different. The  $p$  terms in  $\phi$  are all primitive. All  $b_j$  except for  $j$  a power of  $p$  are decomposable so the circle product of a primitive times any of them is trivial. Thus the only nontrivial term is the one we want.

(b). Again we must confirm that all of our elements are defined in the spectral sequence or space. They are.

(b)(i). We apply  $V^i$  and all we need to do is prove:

$$\sigma ea^{I-\Delta_k}[v^K] \circ a_{(k)} = (-1)^{l(I)-1} \sigma ea^I[v^K]$$

which follows immediately.

(b)(ii). Again we apply  $V^i$  to get down to:

$$\phi(a^{I-\Delta_k}[v^K]) \circ a_{(k+1)} = (-1)^{l(I)} \phi(a^I[v^K]).$$

We now apply  $V_{int}^k$  where  $k = m(I)$  to get

$$\phi(V_{int}^k(a^{I-\Delta_k}[v^K])) \circ a_{(1)} = (-1)^{l(I)} \phi(V_{int}^k(a^I[v^K])).$$

The result now follows like the  $b_{(1)}$  case above.

(e)(i). For  $J \neq 0$  this follows by our induction from (a)(i). If  $I = 0$  we ground our induction by using the definition of the  $b_{(i)}$  which clearly corresponds (up to sign) to  $\gamma_{p^i}(\sigma e)$ . When  $J = 0$  we use (b)(i) and induction while  $I \neq 0$ . The induction starts with  $\sigma e = e \circ e = -b_1 = -b_{(0)}$  (the  $[v^K]$  doesn't matter here).

(e)(ii). For  $J \neq 0$  this follows by our induction from (a)(ii). When  $J = 0$  we use (b)(ii) and induction while  $I \neq 0$ . The induction starts with the recognition that  $\phi([1] - [0])$  is  $a_{(0)}$  and so  $\gamma_{p^i}(\phi([1] - [0]))$  is  $a_{(i)}$  (the  $[v^K]$  doesn't matter here either). These are low degree elements and our space is just  $B\mathbf{Z}/(p)$  in this range so these are easy to see.

(e)(iii). Follows by induction on degree and the definition of the suspension.

(f)(truncated polynomial factor). We show that the even dimensional generators in (f) are all there in  $E_{*,*}^\infty$ . Later, when we are done with the differentials, we finish the proof. All of the elements in (e) must be infinite cycles by the same induction we used to identify them. The elements in (e)(i) correspond to the first term in Lemma 3.6. Because  $(K, J) \in \mathcal{A}_n^+$  in (e)(i), we get  $(K, J) \in \mathcal{A}_n$  in (f). The terms in (e)(i) give us all the terms in (the even part of) (f) with  $m(J) \leq m(I)$ . (e)(ii) corresponds to the fourth part of Lemma 3.6 and gives us the  $m(I) < m(J)$  terms in (f). (e)(iii) corresponds to the elements in the odd part of (f) (which are not hit by differentials).

(c) and (d). The guiding principle for the differentials is that any  $\gamma_{p^i}$  with an  $a_{(n)}$  in it has a differential on it. Of course  $a_{(n)}$  doesn't exist and this is why. We must note that in (c), the elements we assert have differentials are the lowest  $\gamma_{p^i}$  possible because for (c)(i), (e)(i) showed the lower ones were infinite cycles and likewise for (c)(ii) and (e)(ii).

Differentials in our Hopf algebra must start with a generator and go to a primitive, so the only generators which can have differentials are the  $\gamma_{p^i}$  and if they support a nontrivial differential then the target must be an odd degree primitive. All of our odd degree primitives are located in filtration one and are our exterior algebra generators. Note that if  $d^r(\gamma_{p^m}(x)) = y$  where  $y$  is an exterior generator, then it follows formally that (up to nonzero scalar multiplication)

$$d^r(\gamma_{sp^m}(x)) = y\gamma_{(s-1)p^m}(x)$$

for all  $s > 0$ . Thus the factor  $E(y) \otimes \Gamma(x)$  in the  $E^2$ -term gets replaced by

$$TP_1(x, \gamma_p(x), \dots, \gamma_{p^{m-1}}(x))$$

in  $E^{r+1}$ . Each of our differentials takes this form.

Recall that in Lemma 3.6 we have two exterior factors. We want the odd elements in (f) to survive. The remainder are the elements in (d) which we want to be hit by differentials. To show that differentials are what we want them to be we show that the proposed targets must indeed die. We then show, strictly by counting, that the targets are in one-to-one correspondence with our proposed sources. Since our proposed sources are the lowest possible degree elements which could support differentials and our targets are known to be hit, we infer that our proposed sources are in fact our sources.



So, we have two parts left to finish our differentials. (1) We must show the proposed targets are hit by differentials and (2) we must do the counting argument. We defer the counting argument until later.

We want to show that all of the proposed targets,  $\sigma a^I[v^K]b^J$  with  $i_0 = 0$  such that  $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$ , must be zero modulo the same type of elements in  $\mathcal{A}_n^+$ . If they are not zero then they must be represented by  $ea^I[v^K]b^J$  and this, in turn, gives rise to an element  $\sigma ea^I[v^K]b^J$  in the next spectral sequence for the next space. This element is even degree so must represent  $a^I[v^K]b^{J+\Delta_0}$ . Because  $(K, J) \notin \mathcal{A}_n^+$  we have  $(K, J + \Delta_0) \notin \mathcal{A}_n$ . This means that it can be rewritten in terms of  $n$ -allowable elements. The algorithm does not affect the  $i_0 = 0$  condition or the fact that there must be a  $b_{(0)}$ . Thus it can be rewritten in terms of elements that we are working modulo. Thus we know that there is a relation somewhere. There are only two ways for a relation to come up: (1) this last element could be a  $p^{\text{th}}$  power or (2) our differential is as claimed. We are done if we show this last element is not a  $p^{\text{th}}$  power. This follows immediately from Corollary 2.4 which says that a  $p^{\text{th}}$  power must have  $i_0 = 1$ .

We must now do our counting. When that is complete, we'll see that the elements in (f) are all that remain, both odd and even. So, showing our sources are in one-to-one correspondence with our targets will finish the proof of Theorem 3.7. We have to do a similar counting argument in order to solve all the extension problems to get Theorems 1.3 and 1.4 so we separate out the common part.

**Lemma 3.8.** *There is a one-to-one correspondence between the set*

$$\{[v^{K+\Delta_n}]b^{J+(p^n-1)\Delta_0} : (K, J) \in \mathcal{A}_n, j_0 = 0\}$$

and the set

$$\{[v^{K'}]b^{J'} : (K', J') \in \mathcal{A}_n - \mathcal{A}_n^+\}.$$

*Proof.* To see this, write

$$J = p^n \Delta_{d_n} + p^{n+1} \Delta_{d_{n+1}} + \cdots + p^m \Delta_{d_m} + J''$$

where  $m$  is maximal (this can be vacuous, i.e.  $J = J''$ , in which case we set  $m = n - 1$ ) and  $d_n \leq d_{n+1} \leq \cdots \leq d_m$  and  $J''$  is non-negative. Now let

$$J' = J'' + (p^n - 1)\Delta_0 + p^{n+1}\Delta_{d_n-1} + p^{n+2}\Delta_{d_{n+1}-1} + \cdots + p^{m+1}\Delta_{d_m-1}$$

and  $K' = K + \Delta_{m+1}$ . □

We will now do the counting argument for the differentials. We recall that we have to show that the sources listed in Theorem 3.7 (c) are in one-to-one correspondence with the  $ea^I[v^K]b^J$  which have  $i_0 = 0$  and have  $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$ . Strictly for the purposes of counting we introduce a non-element,  $a_{(n)}$ , and incorporate it into our notation,  $e^\varepsilon a^I[v^K]b^J$ . We can, for counting purposes only, identify the set of differential source elements in Theorem 3.7 (c) with the set,

$$\{a^I[v^K]b^J : (K, J) \in \mathcal{A}_n, i_n = 1, i_0 = 0, j_0 = 0\}.$$

Theorem 3.7 (c)(i) gives those with  $m(I) \geq m(J)$  by looking at Theorem 3.7 (e)(i) and Theorem 3.7 (c)(ii) gives those with  $m(I) < m(J)$  by looking at Theorem 3.7 (e)(ii). That is a one-to-one correspondence. We now want to pair these up with the targets of the differentials. To do that we just replace the  $a_{(n)}$  with  $e[v_n]b_{(0)}^{p^n-1}$ .

This gives us a one-to-one correspondence between the sources of the differentials listed in Theorem 3.7 (c) and the set

$$\{ea^I[v^{K+\Delta_n}]b^{J+(p^n-1)\Delta_0} : (K, J) \in \mathcal{A}_n, i_n = 0, i_0 = 0, j_0 = 0\}.$$

Then Lemma 3.8 finishes our counting argument.  $\square$

*Proof of Theorems 1.3 and 1.4 for mod  $p$  homology.* To finish Theorem 1.3 we must solve the extension problems remaining in Theorem 3.7 (f). Theorem 1.4 (a) can just be read off the spectral sequence. The first part of (c), that all  $p^{\text{th}}$  powers are primitive, follows from Corollary 2.4 which says that a  $p^{\text{th}}$  power is divisible by  $a_{(0)}$ . In order to complete the proof we must show that all generators and iterated  $p^{\text{th}}$  powers with an  $a_{(n-1)}$  in them must have nontrivial  $p^{\text{th}}$  powers. (b) will follow and so will the solution to the extensions that we need.

First we'll do a counting argument to show that things can work out the way we suggest. The elements we want to be  $p^{\text{th}}$  powers are the  $a^I[v^K]b^J$  with  $i_0 = 1$  and  $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$ . We need our correspondence to be with generators and  $p^{\text{th}}$  powers with  $i_{n-1} = 1$  in this degree divided by  $p$ . That combination consists of all  $a^{I'}[v^{K'}]b^{J'}$  with  $i_{n-1} = 1$  with  $(K', J') \in \mathcal{A}_n$  (by induction!). First we show that we have a one-to-one correspondence and then we show that our suggested  $p^{\text{th}}$  powers must indeed be  $p^{\text{th}}$  powers. Since we show our elements to be in one-to-one correspondence with the only elements that can possibly have nontrivial  $p^{\text{th}}$  powers, the  $p^{\text{th}}$  powers must be as claimed.

To do our counting we just take the  $p^{\text{th}}$  power of  $a^{I'}[v^{K'}]b^{J'}$ . By Lemma 2.3 we have

$$(a^{I'}[v^{K'}]b^{J'})^{*p} = \pm a^{\Delta_0+s(I'-\Delta_{n-1})}[v^{K'+\Delta_n}]b^{(p^n-1)\Delta_0+s(J')}.$$

Lemma 3.8 now gives us the one-to-one correspondence we seek.

We now have to finish the proof of Theorem 1.4 (c) that the iterated  $p^{\text{th}}$  powers of the generators in Theorem 1.3 are those primitives  $a^I[v^K]b^J$  with  $i_0 = 1$  and  $(K, J) \in \mathcal{A}_n - \mathcal{A}_n^+$  modulo the vector space generated by  $a^I[v^K]b^J$  with  $i_0 = 1$  and  $(K, J) \in \mathcal{A}_n^+$ . Having done our counting argument, it is enough to show that these elements must indeed be  $p^{\text{th}}$  powers. We observe that all of our differentials start in total degrees divisible by  $2p$  so the targets must be in degrees equal to  $-1 \pmod{2p}$ . If our elements are not  $p^{\text{th}}$  powers, then they will suspend to exterior generators in the next bar spectral sequence. Here they cannot be targets of differentials because they have degree  $+1 \pmod{2p}$ . Thus, such an element would suspend once more and be represented by  $a^I[v^K]b^{J+\Delta_0}$  and  $(K, J + \Delta_0)$  would cease to be in  $\mathcal{A}_n$ . It is now in degree  $2 \pmod{2p}$  so it cannot be a  $p^{\text{th}}$  power here which is the only way to create a relation. Thus it must be a  $p^{\text{th}}$  power where we said it would be.  $\square$

## REFERENCES

- [Baa73] N. A. Baas. On bordism theory of manifolds with singularities. *Math. Scand.*, 33:279–302, 1973.
- [BJW95] J. M. Boardman, D. C. Johnson, and W. S. Wilson. Unstable operations in generalized cohomology. In I. M. James, editor, *The Handbook of Algebraic Topology*, chapter 15, pages 687–822. Elsevier, 1995.
- [BM71] N. A. Baas and I. Madsen. On the realization of certain modules over the Steenrod algebra. *Math. Scand.*, 31:220–224, 1971.
- [Boa95] J. M. Boardman. Stable operations in generalized cohomology. In I. M. James, editor, *The Handbook of Algebraic Topology*, chapter 14, pages 585–686. Elsevier, 1995.
- [BW] J. M. Boardman and W. S. Wilson.  $k(n)$ -torsion-free  $H$ -spaces and  $P(n)$ -cohomology. In preparation.

- [ETW] P. J. Eccles, P. R. Turner, and W. S. Wilson. On the Hopf ring for the sphere. *Mathematische Zeitschrift*. To appear.
- [Har91] S. Hara. The Hopf rings for connective Morava  $K$ -theory and connective complex  $K$ -theory. *Journal of Mathematics of Kyoto University*, 31(1):43–70, 1991.
- [HRW] M. J. Hopkins, D. C. Ravenel, and W. S. Wilson. Morava Hopf algebras and spaces  $K(n)$  equivalent to finite Postnikov systems. Submitted.
- [JW75] D. C. Johnson and W. S. Wilson. BP-operations and Morava’s extraordinary  $K$ -theories. *Mathematische Zeitschrift*, 144:55–75, 1975.
- [Kra90] R. Kramer. *The periodic Hopf ring of connective Morava  $K$ -theory*. PhD thesis, Johns Hopkins University, 1990.
- [Mor79] J. Morava. A product for odd-primary bordism of manifolds with singularities. *Topology*, 18:177–186, 1979.
- [RW77] D. C. Ravenel and W. S. Wilson. The Hopf ring for complex cobordism. *Journal of Pure and Applied Algebra*, 9:241–280, 1977.
- [RW80] D. C. Ravenel and W. S. Wilson. The Morava  $K$ -theories of Eilenberg-Mac Lane spaces and the Conner-Floyd conjecture. *American Journal of Mathematics*, 102:691–748, 1980.
- [Str92] N. Strickland. *Bott periodicity and Hopf rings*. PhD thesis, University of Manchester, 1992.
- [SY76] N. Shimada and N. Yagita. Multiplications in complex bordism theory with singularities. *Publications of Research Institute of Mathematical Sciences, Kyoto University*, 12:259–293, 1976.
- [Tur] P. R. Turner. Dickson coinvariants and the homology of  $H_*QS^0$ . *Mathematische Zeitschrift*. To appear.
- [TW80] R. W. Thomason and W. S. Wilson. Hopf rings in the bar spectral sequence. *Quarterly Journal of Mathematics*, 31:507–511, 1980.
- [Wil75] W. S. Wilson. The  $\Omega$ -spectrum for Brown-Peterson cohomology, Part II. *American Journal of Mathematics*, 97:101–123, 1975.
- [Wil82] W. S. Wilson. *Brown-Peterson homology: an introduction and sampler*. Number 48 in C.B.M.S. Regional Conference Series in Mathematics. American Mathematical Society, Providence, Rhode Island, 1982.
- [Wil84] W. S. Wilson. The Hopf ring for Morava  $K$ -theory. *Publications of Research Institute of Mathematical Sciences, Kyoto University*, 20:1025–1036, 1984.
- [Wür77] U. Würzler. On products in a family of cohomology theories associated to the invariant prime ideals of  $\pi_*(BP)$ . *Commentarii Mathematici Helvetici*, 52:457–481, 1977.
- [Yag77] N. Yagita. On the algebraic structure of cobordism operations with singularities. *Journal of the London Mathematical Society*, 16:131–141, 1977.

UNIVERSITY OF ROCHESTER, ROCHESTER, NEW YORK 14627  
*E-mail address:* `drav@troi.cc.rochester.edu`

JOHNS HOPKINS UNIVERSITY, BALTIMORE, MARYLAND 21218  
*E-mail address:* `wsw@math.jhu.edu`