Practice Final Exam Solutions, Linear Algebra (110.201), Spring, 2021, W. Stephen Wilson

Name : ____________________________________

TA Name and section: ____________________________________

Open book.

You can print out the exam and work it and then upload it, or you can work on your own paper and upload it. Just be very clear about what you are doing if you go that way. You must show your work.

1. (2 points) Let

$$C = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

Solve $CX = 0$.

Not much computation is necessary to put this in reduced row echelon form. just subtract row one from row 4 and row 2 from row 3.

$$rref(C) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

So we can read off $x_1 = -x_4$ and $x_2 = -x_3$. So

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}}$$

But, as usual, it is wise to check it.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

2. (2 points) Find an orthonormal basis for the kernel of $C : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

We have, from problem 1, a basis for the kernel. The basis elements are perpendicular without any need to compute. All we have to do is divide by the length, which is easily seen to be $\sqrt{2}$ in both cases. So, the answer is:

$$\boxed{\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}}$$

The misfortune here is that there are an infinite number of different possible orthonormal bases for the kernel. These are the most obvious ones. The most likely variations are to change a sign on a vector or switch the two vectors. That makes for 8 possibilities right there.

3. (2 points) Let

$$B = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}.$$

Solve $BX = 0$.

We can just read off $rref(B)$, all we have to do is add the first row to the last and the second to the third and then multiply first and second by -1.

$$rref(B) = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

From this we can read off the solution

$$\boxed{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}}$$

And, of course, we really do have to check this. If we don't, that will be the time we make a mistake.

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

4. (2 points) Find an orthonormal basis for the kernel of $B : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$.

We have, from problem 3, a basis for the kernel. The basis elements are perpendicular without any need to compute. All we have to do is divide by the length, which is easily seen to be $\sqrt{2}$ in both cases. So, the answer is:

$$\boxed{\left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}}$$

5. (2 points) Find the characteristic polynomial for the matrix C of problem 1, i.e. the determinant of $C - \lambda I$.

We need the determinant of

$$\begin{pmatrix} 1-\lambda & 0 & 0 & 1 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 1-\lambda \end{pmatrix}.$$

Without changing the determinant, we can take $(1-\lambda)$ times the fourth row and subtract if from the first. Likewise, we can do this with the 3rd and 2nd rows, to get:

$$\begin{pmatrix} 0 & 0 & 0 & 1-(1-\lambda)^2 \\ 0 & 0 & 1-(1-\lambda)^2 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 1 & 0 & 0 & 1-\lambda \end{pmatrix}.$$

We can now switch rows 1 and 4 and also 2 and 3. Each changes the sign once, so there is no change with the determinant to get

$$\begin{pmatrix} 1 & 0 & 0 & 1-\lambda \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 1-(1-\lambda)^2 & 0 \\ 0 & 0 & 0 & 1-(1-\lambda)^2 \end{pmatrix}.$$

This is an upper triangular matrix so the determinant is just the product of the diagonal terms

$$\boxed{[1-(1-\lambda)^2]^2 = [1-(1-2\lambda+\lambda^2)]^2 = [2\lambda-\lambda^2]^2 = \lambda^4 - 4\lambda^3 + 4\lambda^2 = \lambda^2(\lambda-2)^2}$$

6. (2 points) With multiplicity, find the 4 eigenvalues for C, and write them $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$.

We can just read this off from problem 5.

$$\boxed{2 \geq 2 \geq 0 \geq 0}$$

7. (2 points) For each λ_i in the previous problem, find an eigenvector. Write them in the same order as the λ_i in such a way that the eigenvectors you find form an orthonormal basis for $\mathbb{R}^4$.

The matrix B of problem 3 is the matrix for computing the eigenvectors for the eigenvalue 2. Problem 4 gives an orthonormal basis for the eigenspace for 2. Similarly, the matrix C of problems 1 and 2 gives the orthonormal basis for the eigenvalue 0. We can just write down the answer here from these results.

$$\boxed{\text{for} \quad 2 \quad \text{we have} \quad \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\} \qquad \text{for} \quad 0 \quad \text{we have} \quad \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}}$$

8. (4 points) Find an orthogonal matrix S such that $S^T C S$ is

$$\begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{pmatrix}.$$

All we have to do here is combine the vectors in the previous problem to make the columns of S, so

$$S = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}$$

It is only fair that we check this result since we can.

$$S^TCS = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{2}{\sqrt{2}} & 0 & 0 \\ \frac{2}{\sqrt{2}} & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

9. (2 points) Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

Find the singular values for $A : \mathbb{R}^4 \longrightarrow \mathbb{R}^2$, and write them $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \sigma_4 \geq 0$

To compute the singular values, we find the eigenvalues of A^TA, which is

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

This is our matrix C and we already know the eigenvalues: 2,2,0,0. So, our singular values are the square roots:

$$\boxed{\sqrt{2} \geq \sqrt{2} \geq 0 \geq 0 \geq 0}$$

10. (6 points) Find an orthonormal basis for $\mathbb{R}^4$, $\{u_1, u_2, u_3, u_4\}$ and an orthonormal basis for $\mathbb{R}^2$, $\{w_1, w_2\}$, such that $Au_1 = \sigma_1 w_1$, $Au_2 = \sigma_2 w_2$, and u_3 and u_4 are in the kernel, i.e. $Au_3 = 0 = Au_4$.

In problem 7 we already found the u_i, they are the eigenvectors for C.

$$\boxed{\{u_1, u_2, u_3, u_4\} = \left\{ \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{-1}{\sqrt{2}} \end{pmatrix}, \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix} \right\}}$$

We compute

$$Au_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{2}} \\ 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \sqrt{2} w_1 = \sigma_1 w_1$$

We compute

$$Au_2 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{2}} \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sqrt{2} w_2 = \sigma_2 w_2$$

So we see that

$$\boxed{\{w_1, w_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}}$$

11. (6 points) Find orthogonal matrices U and V and a matrix Σ so that $A = U\Sigma V^T$ as in the singular value decomposition theorem. Write your answer out as the product, $U\Sigma V^T$.

The matrix V is just the matrix S we computed in problem 8, so we need to take its transpose. U is just the identity made from the w_i just found in the previous problem. Thus:

$$\boxed{\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix}}$$

Which is easily checked.

12. (6 points) Consider the quadratic form

$$q(x_1, x_2, x_3, x_4) = x_1^2 + x_2^2 + 2x_1x_4 + 2x_2x_3 + x_3^2 + x_4^2$$

Change the basis so that you have it written in the form

$$q(y_1, y_2, y_3, y_4) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 + \lambda_4 y_4^2$$

where we want $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$.

Explain your work, but all we want for an answer is the new formula above (and the work, or, if you don't have to work, an explanation).

The symmetric matrix for this quadratic form is just our matrix C from problem 1, so we have $q(X) = X^T C X$. We have already found the eigenvalues and eigenvectors for this in problem 7. So see problem 7 for the answers. So, we have

$$\boxed{q(y_1, y_2, y_3, y_4) = 2y_1^2 + 2y_2^2 + 0y_3^2 + 0y_4^2 = 2y_1^2 + 2y_2^2}$$

13. (6 points) Set $q(y_1, y_2, y_3, y_4) = 2$. Find the set of points in $\mathbb{R}^4$ closest to the origin. Give your answer in terms of your new coordinates. (4 points) How far is the closest point from the origin? (2 points)

In terms of our new coordinates, this is just $2y_1^2 + 2y_2^2 = 2$, or $\boxed{y_1^2 + y_2^2 = 1}$. To get the points closest to the origin, we use $\boxed{y_3 = y_4 = 0}$. They are arbitrary, but the points closest will be these. What is left is just a unit circle. So, all the points are the circle are closest and $\boxed{\text{they are all distance 1}}$ from the origin.

14. (4 points) Set $q(y_1, y_2, y_3, y_4) = 2$. Now, consider only those points with $y_4 = 0$. There are now just 3 coordinates, so we can think of ourselves in $\mathbb{R}^3$. Describe the geometric object these equations give.

With $y_4 = 0$, in 3-space, we have the equation $y_1^2 + y_2^2 = 1$. That is a circle, but the coordinate y_3 is arbitrary, so for every $y_3 \in \mathbb{R}$, we get a circle of radius 1. In other words, we have $\boxed{\text{an infinite circular tube.}}$

15. (3 points) Let

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Calculate the singular values for F as $\sigma_1 \geq \sigma_2 \geq 0$.

To make the computations, we need $F^T F$, i.e.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

We immediately see that the eigenvalues for this are 2 and 2 and the corresponding

$\boxed{\text{singular values are both } \sqrt{2}}$.

16. (6 points) Find an orthonormal basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ and an orthonormal basis $\{w_1, w_2, w_3, w_4\}$ for $\mathbb{R}^4$ such that $Fu_1 = \sigma_1 w_1$ and $Fu_2 = \sigma_2 w_2$.

Orthonormal eigenvectors for $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ are

$$\boxed{u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } u_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

To find w_1 and w_2 we compute $Fu_1 = \sqrt{2}w_1$ and $Fu_2 = \sqrt{2}w_2$.

$$Fu_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \sigma_1 w_1$$

$$Fu_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \sigma_2 w_2$$

$\boxed{\text{Our } w_1 \text{ and } w_2 \text{ are just as in problem 7 so we can use}}$

$\boxed{\text{the other two basis vectors from problem 7 for } w_3 \text{ and } w_4.}$

17. (4 points) Write F as the product of 3 matrices in the form of the singular value decomposition.

We need to write $F = U\Sigma V^T$ where U and V are orthogonal and Σ is made from the singular values. The vectors w_i give U, which just happens to be the matrix S from problem 11. The matrix V is just $I_2 = I_2^T$. We get:

$$\boxed{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

Note that the easy way to do this is to just take the transpose of the solution to problem 11.