201 Linear Algebra, Practice Midterm2 Solutions

1. $\mathcal{B} = \{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \}$.

$$A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$A \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 3 \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1/2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 5/2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

$$A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} = 3/2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + 1/2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 1/2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

The $\mathcal{B}$-matrix for the transformation defined by $T(\vec{x}) = A\vec{x}$ is

$$B = \begin{pmatrix} 1 & 1/2 & 3/2 \\ 0 & 1/2 & 1/2 \\ 2 & 5/2 & 1/2 \end{pmatrix}$$

so that $B[\vec{x}]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}}$ for all $\vec{x} \in \mathbb{R}^3$.

Alternatively, $B = S^{-1}AS$ where $S = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ is the change of basis matrix $S_{\mathcal{B} \to \mathcal{S}}$.

2. (a) $\mathcal{S} = \{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \}$ is the standard basis for $\mathbb{R}^{2\times 2}$.

$$T(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$T(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

$$T(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}$$

$$T(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 1 \end{bmatrix}$$

Therefore the $\mathcal{S}$-matrix for T is

$$A = \begin{pmatrix} 1 & 2 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

i.e. for any $N \in \mathbb{R}^{2\times 2}$, $A[M]_{\mathcal{S}} = \left[M \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right]_{\mathcal{S}}$.

(b) The $\mathcal{S}$-matrix for T, A row reduces to I_4. Therefore A is invertible. This means T is an invertible linear transformation. $T^{-1}(M) = M \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}^{-1}$.

Therefore $\text{Ker}T = \{\vec{0}\} = \{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\}$. Also $\text{Im}T = \mathbb{R}^{2\times 2}$ by rank nullity theorem.

(c) Let B be the $\mathcal{B}$-matrix for T, i.e $B[M]_{\mathcal{B}} = [T(M)]_{\mathcal{B}}$. The column vectors of B are as follows $\left[T(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix})\right]_{\mathcal{B}}, \left[T(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix})\right]_{\mathcal{B}}, \left[T(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix})\right]_{\mathcal{B}}$ and $\left[T(\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix})\right]_{\mathcal{B}}$.

Alternatively, $B = S^{-1}AS$ where S is the change of basis matrix $S_{\mathcal{B}\to\mathcal{S}} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$.

Compute S^{-1} and then B.

(d) See (c)

3. This subspace $V = \{\vec{x} \in \mathbb{R}^4 \mid \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix} \bullet \vec{x} = 0\} = \text{Ker}\begin{pmatrix} 1 & 2 & 0 & 1 \end{pmatrix} = \{\begin{pmatrix} -2s-u \\ s \\ t \\ u \end{pmatrix} \mid s,t,u \in \mathbb{R}\} =$ $\text{Span}\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}$. This is a basis for V.

Apply Gram-Schmidt to this basis.

4. The least-squares solution to this solves the equation

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}\vec{x}_* = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\vec{x}_* = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$$

5. (a) FALSE. The zero transformation that takes every polynomial to the zero 2×2 matrix is not invertible.

(b) TRUE. Suppose $T : V \to V$ is a linear transformation. $\mathcal{B}$ a basis for the n-dimensional vector space V. The $\mathcal{B}$-matrix B for T defines the linear transformation $B : \mathbb{R}^n \to \mathbb{R}^n$ that fits into the following commutative diagram

$$\begin{array}{ccc} V & \xrightarrow{T} & V \\ {\scriptstyle L_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle L_{\mathcal{B}}} \\ \mathbb{R}^n & \xrightarrow{B} & \mathbb{R}^n \end{array}$$

Therefore T is a composition of invertible linear transformations $T = L_{\mathcal{B}} \circ B \circ L_{\mathcal{B}}^{-1}$, hence T is invertible.

(c) TRUE. Let the line be $L = \text{Span}\{\vec{u}\}$. Let $\mathcal{B} = \{\text{Rot}_{45^\circ}\vec{u}, \text{Rot}_{-45^\circ}\vec{u}\}$. Then the $\mathcal{B}$-matrix for the reflection map is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(d) FALSE. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is orthogonal. A is the standard basis matrix for the reflection about the line $y = x$. If $\mathcal{B} = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$, the $\mathcal{B}$-matrix for the same transformation is $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, which is not orthogonal.

(e) TRUE. A preserves lengths. Therefore $||\vec{x}|| = ||A\vec{x}|| = ||AA\vec{x}|| = ||AAA\vec{x}||$.