LINEAR COORDINATES

1. Bases of $\mathbb{R}^n$

There are many different bases of $\mathbb{R}^n$. We have already discussed the *standard basis*

$$\vec{e}_1, \dots, \vec{e}_n$$

where here $\vec{e}_i$ is the vector with ith entry 1 and all others 0. If $\vec{v}_1, \cdots, \vec{v}_n$ is another basis, then

$$S = \begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_n \end{bmatrix}$$

must have

(1) $\ker(S) = \left\{ \vec{0} \right\}$ which corresponds to $\vec{v}_1, \dots, \vec{v}_n$ being linearly independent

(2) $\mathrm{Im}\,(S) = \mathbb{R}^n$ which corresponds to the vectors spanning all of $\mathbb{R}^n$.

and/or . In particular, $\vec{v}_1, \cdots, \vec{v}_n$ is a basis $\iff$ S is an invertible $n \times n$ matrix.

EXAMPLE: $\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a basis of $\mathbb{R}^2$. As such the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ can be expressed as a linear combination of this basis. The matrix

$$S = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$$

satisfies

$$\mathrm{rref}[S|I_2] = \begin{bmatrix} 1 & 0 & | & -1 & -2 \\ 0 & 1 & | & -1 & -1 \end{bmatrix}$$

and so S is invertible and we have a verified that the pair of vectors is a basis of $\mathbb{R}^2$. We wish to write

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = S \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

for some choice of c_1, c_2. Clearly,

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S^{-1} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

So

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 1 \\ -1 \end{bmatrix} - 3 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

2. Linear Coordinates

Given a subspace W of dimension m we wish to "parameterize" W by $\mathbb{R}^m$. That is to associate m numbers (called coordinates) to each vector in W that uniquely determines the vector. For a general subspace this requires making some choice of basis of W. With that in mind, for $W \subset \mathbb{R}^n$ a subspace, let $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ be an *(ordered) basis* of W.

EXAMPLE: $(\vec{e}_1, \vec{e}_2)$ and $(\vec{e}_2, \vec{e}_1)$ are different ordered bases of $\mathbb{R}^2$.

Any $\vec{w} \in W$ can be *uniquely* written as

$$\vec{w} = c_1 \vec{v}_1 + \cdots + c_m \vec{v}_m$$

The existence of the c_i is due to the fact that the basis spans. The uniqueness is due to the fact that it is linearly independent. Specifically, if

$$\vec{x} = c_1\vec{v}_1 + \cdots + c_m\vec{v}_m = d_1\vec{v}_1 + \cdots + d_m\vec{v}_m,$$

then the linear relation

$$(c_1 - d_1)\vec{v}_1 + \cdots + (c_m - d_m)\vec{v}_m = \vec{0}$$

holds. As the vectors are linearly independent, this is trivial and hence $c_i = d_i$ as desired.

The scalars $c_1, \ldots, c_m$ are called the *$\mathcal{B}$-coordinates of $\vec{w}$*. Putting thise together gives the vector

$$[\vec{w}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} \in \mathbb{R}^m$$

is the *$\mathcal{B}-$coordinate vector of $\vec{x}$.*

EXAMPLE: If $\mathcal{E} = (\vec{e}_1, \cdots, \vec{e}_n)$ is the standard basis, then

$$[\vec{x}]_{\mathcal{E}} = \vec{x}$$

.

Observe that

$$\vec{x} = \underbrace{\begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_m \end{bmatrix}}_{S}[\vec{x}]_{\mathcal{B}}$$

We will call the matrix

$$S = \begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_m \end{bmatrix}$$

a *change of basis matrix.* It takes the standard basis of $\mathbb{R}^m$ to the basis $\mathcal{B}$ of W.

EXAMPLE: If $\mathcal{B} = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix} \right)$, are from the previous section, then

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

If $\mathcal{B}' = \left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$ are the same basis with the opposite ordering, then

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}_{\mathcal{B}'} = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$$

If $[\vec{y}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$\vec{y} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

while if $[\vec{y}]_{\mathcal{B}'} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, then

$$\vec{y} = 1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + 2\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

EXAMPLE: Find an ordered basis of $\ker(A)$ for

$$A = \begin{bmatrix} 0 & 1 & 2 & 2 \\ 0 & 2 & 4 & 5 \end{bmatrix}.$$

First, compute

$$\operatorname{rref}(A) = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Observe that the free variables are $f_1 = x_1$ and $f_2 = x_3$. Hence, have and ordered basis

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right).$$

If $\vec{z} \in \ker(A)$ and $[\vec{z}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, then

$$\vec{z} = 2 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

EXAMPLE: Find an ordered basis of Im (A) where

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix}$$

First compute,

$$\operatorname{rref}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and conclude from this that first and third columns are pivot columns. Hence, an ordered basis of Im (A) is

$$\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right).$$

Clearly, the fourth column of A,

$$A\vec{e}_4 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \in \operatorname{Im}(A)$$

Moreover,

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \text{ and so } \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

3. Linearity of coordinates

Linear coordinates have natural linearity properties. Indeed, if $\mathcal{B}$ is an (ordered) basis of a subspace $W \subset \mathbb{R}^n$, then

(1) $[\vec{x} + \vec{y}]_{\mathcal{B}} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$ for all $\vec{x}, \vec{y} \in W$.
(2) $[k\vec{x}]_{\mathcal{B}} = k[\vec{x}]_{\mathcal{B}} =$ for all $\vec{x} \in W$ and $k \in \mathbb{R}$.

To see why (1) holds, first let $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_m)$ and write

$$\vec{x} = c_1\vec{v}_1 + \ldots + c_m\vec{v}_m \text{ and } \vec{y} = d_1\vec{v}_1 + \ldots + d_m\vec{v}_m$$

Readily see that,

$$\vec{x} + \vec{y} = (c_1 + d_1)\vec{v}_1 + \cdots + (c_m + d_m)\vec{v}_m.$$

Hence,

$$[\vec{x} + \vec{y}]_{\mathcal{B}} = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_m + d_m \end{bmatrix} = \begin{bmatrix} c_1 \\ \vdots \\ c_m \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} = [\vec{x}]_{\mathcal{B}} + [\vec{y}]_{\mathcal{B}}$$

as claimed. (2) follows in the same fashion.

EXAMPLE: Let $\mathcal{B} = \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right)$ be an ordered basis of $\mathbb{R}^2$. Find $[\vec{e}_1]_{\mathcal{B}}$ and $[\vec{e}_2]_{\mathcal{B}}$ and use this to find $[\vec{x}]_{\mathcal{B}}$ for general $\vec{x}$. By inspection, $\vec{e}_1 = 2\begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so

$$[\vec{e}_1]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Likewise,

$$\vec{e}_2 = 2\begin{bmatrix} 3 \\ 2 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and so

$$[\vec{e}_2]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}.$$

A general $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ then hass

$$[\vec{x}]_{\mathcal{B}} = x_1\begin{bmatrix} 2 \\ -1 \end{bmatrix} + x_2\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

4. Matrix of a linear transformation

Consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ and an (ordered) basis $\mathcal{B} = (\vec{v}_1, \dots, \vec{v}_n)$ of $\mathbb{R}^n$ with change of basis matrix

$$S = \begin{bmatrix} \vec{v}_1 & | & \cdots & | & \vec{v}_n \end{bmatrix}.$$

There exists a unique $n \times n$ matrix $[T]_{\mathcal{B}}$ so that

$$[T(\vec{x})]_{\mathcal{B}} = [T]_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$$

We call this matrix *the $\mathcal{B}$-matrix of T*. One explicitly has

$$[T]_{\mathcal{B}} = \begin{bmatrix} [T(\vec{v}_1)]_{\mathcal{B}} & | & \cdots & | & [T(\vec{v}_n)]_{\mathcal{B}} \end{bmatrix}.$$

That is, the columns of $[T]_{\mathcal{B}}$ are precisely the $\mathcal{B}$-coordinate vectors of the images under T of the ordered basis vectors.

EXAMPLE: If $\mathcal{E} = (\vec{e}_1, \dots, \vec{e}_n)$ is the standard basis, then

$$[T]_{\mathcal{E}} = [T].$$

Call $[T]$ the *standard matrix* of T.

EXAMPLE: Suppose $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $T(\vec{x}) = A\vec{x}$ and $\mathcal{B} = (\vec{e}_2, \vec{e}_1)$. One computes,

$$[T(\vec{e}_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } [T(\vec{e}_1)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Hence,

$$B = [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

In this case, the change of basis matrix is

$$S = \begin{bmatrix} \vec{e}_2 & \mid & \vec{e}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Together these form a "diagram"

$$\begin{array}{ccccc} \vec{x} & \xrightarrow{A} & A\vec{x} & & = T(\vec{x}) \\ {\scriptstyle S}\uparrow & & {\scriptstyle S}\uparrow & & \\ [\vec{x}]_{\mathcal{B}} & \xrightarrow{B} & B[\vec{x}]_{\mathcal{B}} & & = [T(\vec{x})]_{\mathcal{B}} \end{array}$$

This is understood to mean that

$$A(S[\vec{x}]_{\mathcal{B}}) = S(B([\vec{x}]_{\mathcal{B}})) = T(\vec{x}).$$

As $[\vec{x}_{\mathcal{B}}$ is arbitrary, this really means $AS = SB$ as matrices. Indeed, one readily checks

$$AS = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

while

$$SB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

More generally, let $T : \mathbb{R}^n \to \mathbb{R}^n$ be linear and suppose $\mathcal{B} = (\vec{v}_1, \ldots, \vec{v}_n)$ is an (ordered) basis. If $S = \begin{bmatrix} \vec{v}_1 & \mid & \cdots & \mid & \vec{v}_n \end{bmatrix}$ is the change of basis matrix of $\mathcal{B}$, then the following equivalent identities hold

(1) $[T]S = S[T]_{\mathcal{B}}$,
(2) $[T] = S[T]_{\mathcal{B}}S^{-1}$, and
(3) $[T]_{\mathcal{B}} = S^{-1}[T]S$.

EXAMPLE: Consider

$$T(\vec{x}) = A\vec{x}, A = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find a basis $\mathcal{B}$ so that

$$[T]_{\mathcal{B}} = \begin{bmatrix} \hat{A} & \mid & \vec{0} \end{bmatrix}$$

where $\hat{A}$ is 2×3. Idea: Find an ordered basis whose last vector is a non-zero member of $\ker(A)$. Can see

$$\ker(A) = \operatorname{span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right).$$

So want $\mathcal{B}$ to have its third vector be

$$\vec{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}.$$

Need two more vectors to get a basis we have a lot of freedom in this choice. By inspection, can take $\vec{v}_1 = \vec{e}_1$ and $\vec{v}_2 = \vec{e}_3$ and so $\mathcal{B} = (\vec{v}_1, \vec{v}_2, \vec{v}_3)$ has change of basis matrix

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

One readily computes that

$$S^{-1} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

Hence,

$$[T]_{\mathcal{B}} = S^{-1}[T]S = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Observe $\hat{A} = \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$ and has $rank(\hat{A}) = 2$ so $\ker(\hat{A}) = \left\{\vec{0}\right\}$.