KERNEL AND IMAGE

1. Motivating problem

Let

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$$

we want to consider the following questions:

(1) For which $\vec{y} \in \mathbb{R}^3$ does the system $A\vec{x} = \vec{y}$ have some solution?
(2) What are all solutions $A\vec{x} = \vec{y}$ for a fixed $\vec{y} \in \mathbb{R}^3$?

First we use Gauss-Jordan elimination to compute:

$$\text{rref}[A|\vec{y}] = \text{rref} \begin{bmatrix} 1 & 2 & 0 & | & y_1 \\ 1 & 1 & 1 & | & y_2 \\ 2 & 3 & 1 & | & y_3 \end{bmatrix} = \text{rref} \begin{bmatrix} 1 & 0 & 2 & | & -y_1 + 2y_2 \\ 0 & 1 & -1 & | & y_1 - y_2 \\ 0 & 0 & 0 & | & y_3 - y_1 - y_2 \end{bmatrix}$$

On the one hand, if $y_3 - y_1 - y_2 = 0$ then this is in RREF and we are done, on the other hand, if $y_3 - y_1 - y_2 \neq 0$, then we will have a pivot in final column. In other words, there is a solution if and only if

$$\vec{y} \in \left\{ \begin{bmatrix} s \\ t \\ s+t \end{bmatrix} s, t \in \mathbb{R} \right\} := \text{Im}\,(A)$$

. Geometrically, $\text{Im}\,(A)$ is a plane through the origin. Now, fix $\vec{y} \in \text{Im}\,(A)$ (so $y_3 = y_1 + y_2$) and take the free variable $x_3 = t$. This implies

$$A\vec{v} = \vec{y} \iff \vec{v} \in \left\{ \begin{bmatrix} -y_1 + 2y_2 - 2t \\ y_1 - y_2 + t \\ t \end{bmatrix} : t \in \mathbb{R} \right\}.$$

Geometrically, this is a line parallel to the vector

$$\vec{w} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ which goes through } \begin{bmatrix} -y_1 + 2y_2 \\ y_1 - y_2 \\ 0 \end{bmatrix}.$$

Observe, $A\vec{w} = \vec{0}$, i.e., $\vec{w}$ is a non-zero solution for $\vec{y} = \vec{0}$. Consider,

$$\left\{ t \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\} = \ker(A).$$

What we have already worked out means

$$A\vec{v} = \vec{y} \iff \vec{v} \in \left\{ \begin{bmatrix} -y_1 + 2y_2 \\ y_1 - y_2 \\ 0 \end{bmatrix} + \vec{w} : \vec{w} \in \ker(A) \right\}.$$

That is, to find all solutions to $A\vec{x} = \vec{y}$ just need to find *one* solution. All other solutions obtained from it by adding an element of $\ker(A)$. As such we can answer our original questions:

(1) There is a solution only when $\vec{y} \in \text{Im}\,(A)$, i.e., the vector $\vec{y}$ lies on a plane through the origin in $\mathbb{R}^3$ determined by A.
(2) If there is a solution to $A\vec{x} = \vec{y}$, then the set of all solutions consist of some line in $\mathbb{R}^3$ and all of these lines are parallel to $\ker(A)$ which is the line through the origin corresponds to the solutions of $A\vec{x} = \vec{0}$.

As we will see this is a general phenomena.

2. Image of a linear transformation/matrix

Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ let

$$\text{Im}\,(T) = \{T(\vec{x}) : \vec{x} \in \mathbb{R}^m\} \subset \mathbb{R}^n$$

be the *image* of T. Similarly, for a $n \times m$ matrix A

$$\text{Im}\,(A) := \text{Im}\,(T_A) = \{A\vec{x} : \vec{x} \in \mathbb{R}^m\}.$$

Call $\text{Im}\,(A)$ the *image* of A or the *column space* of A. Observe that $\vec{y} \in \text{Im}\,(A)$ if and only if the system $A\vec{x} = \vec{y}$ has at least one solution.

EXAMPLE:

$$\text{If } A = \begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}, \text{ then}$$

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_1 - 2x_2 \end{bmatrix} = (x_1 - x_2) \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

As, x_1 and x_2 are arbitrary this means

$$\text{Im}\,(A) = \left\{ t \begin{bmatrix} 1 \\ 2 \end{bmatrix} : t \in \mathbb{R} \right\}$$

is the line through the origin $x_2 = 2x_1$.

EXAMPLE: If A is $n \times n$ and invertible, then $\text{Im}\,(A) = \mathbb{R}^n$. This is because, $A\vec{x} = \vec{y}$ must be solvable (by definition) for all $\vec{y} \in \mathbb{R}^n$.

EXAMPLE: If $A = 0_{n \times m}$ is the $n \times m$ *zero matrix*, then $\text{Im}\,(A) = \left\{\vec{0}\right\}$.

EXAMPLE: Let A be an $n \times p$ matrix and B be a $p \times m$ matrix, then

$$\text{Im}\,(AB) \subset \text{Im}\,(A).$$

Indeed, $\vec{y} \in \text{Im}\,(AB)$ if and only if $\vec{y} = (AB)\vec{x}$ for some $\vec{x} \in \mathbb{R}^m$, but $(AB)\vec{x} = A(B\vec{x}) = \vec{y}$ which means that $\vec{y} \in \text{Im}\,(A)$ as it is the image of $B\vec{x}$ under A.

3. Kernel of a linear transformation/matrix

Given a linear transformation $T : \mathbb{R}^m \to \mathbb{R}^n$ let

$$\ker(T) = \left\{ \vec{x} : T(\vec{x}) = \vec{0} \right\} \subset \mathbb{R}^n$$

be the *kernel* of T. Similarly, for a $n \times m$ matrix A let

$$\ker(A) := \ker(T_A) = \left\{ \vec{x} : A\vec{x} = \vec{0} \right\}.$$

Call $\ker(A)$ the *kernel* of A or the *null space* of A. Observe that $\vec{w} \in \ker(A)$ that if the system $A\vec{x} = \vec{y}$ has solution $\vec{x} = \vec{v}$, then $\vec{x} = \vec{v} + \vec{w}$ is also a solution. A

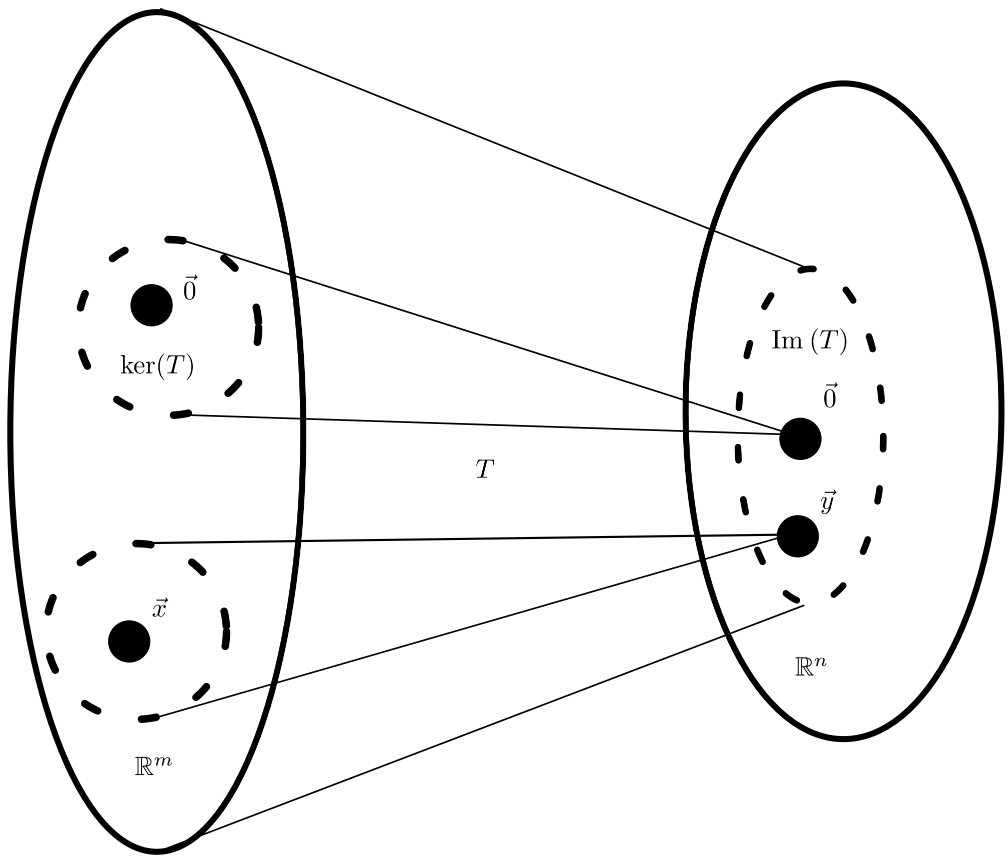

FIGURE 1. A schematic picture of $\ker(T)$, $\mathrm{Im}\,(T)$ and points $\vec{y} = T(\vec{x})$ for a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$.

consequence of this is that if $\ker(A) = \left\{\vec{0}\right\}$, then the system $A\vec{x} = \vec{y}$ can have at most one solution.

EXAMPLE: Find $\ker(T)$ where $T : \mathbb{R}^4 \to \mathbb{R}^2$ is given by

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -2 & 0 & 2 \end{bmatrix} \vec{x}.$$

First compute

$$\mathrm{rref}([T]) = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \end{bmatrix}.$$

Setting the free variables $x_3 = s$ and $x_4 = t$ obtain,

$$A\vec{x} = \vec{0} \iff \vec{x} = \left\{ \begin{bmatrix} -s-t \\ s \\ t \\ s \end{bmatrix} : s, t \in \mathbb{R} \right\}.$$

EXAMPLE: If A is an $n \times n$ invertible matrix, then $\ker(A) = \left\{\vec{0}\right\}$.

EXAMPLE: If $A = 0_{n\times m}$, then $\ker(A) = \mathbb{R}^m$.

EXAMPLE: If A is $n \times p$ and B is $p \times m$, then

$$\ker(B) \subset \ker(AB).$$

Indeed, if $\vec{x} \in \ker(B)$, then $B\vec{x} = \vec{0}$ and so $(AB)\vec{x} = A(B\vec{x}) = A\vec{0} = \vec{0}$.

EXAMPLE: If $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $ker(A) = \left\{ \begin{bmatrix} t \\ 0 \end{bmatrix} : t \in \mathbb{R} \right\}$. However, $A^2 = 0_{2\times 2}$, so $ker(A^2) = \mathbb{R}^2$.

Let us now summarize some facts about the relationship between the image, the kernel and the rank of a $n \times m$ A matrix (in (4) we assume A is square).

(1) $\mathrm{Im}\,(A) = \left\{\vec{0}\right\} \iff A = 0_{m\times n} \iff \ker(A) = \mathbb{R}^m \iff \mathrm{rank}(A) = 0$.
(2) $\mathrm{Im}\,(A) = \mathbb{R}^n \iff \mathrm{rank}(A) = n$. This is because there is a pivot in every row of $\mathrm{rref}(A)$ if and only if one can always solve $A\vec{x} = \vec{y}$ for any $\vec{y}$.
(3) $\ker(A) = \left\{\vec{0}\right\} \iff \mathrm{rank}(A) = m$. This is because there is a pivot in every column of $\mathrm{rref}(A)$ if and only if $A\vec{x} = \vec{0}$ has only the solution $\vec{x} = \vec{0}$.
(4) $(n = m)$ A is invertible $\iff \ker(A) = \left\{\vec{0}\right\} \iff \mathrm{Im}\,(A) = \mathbb{R}^n$.

To see (4) first note that if A is invertible, then $\ker(A) = \left\{\vec{0}\right\}$ by definition. Conversely, if $\ker(A) = \left\{\vec{0}\right\}$, then $\mathrm{rank}(A) = n$, so as A is square, $\mathrm{Im}\,(A) = \mathbb{R}^n$. Hence, can always solve $A\vec{x} = \vec{y}$ (as A has full image) and also this solution is unique (as $\ker(A) = \{0\}$). This means A is invertible.

EXAMPLE: If A and B are $n \times n$ matrices and $BA = I_n$, then $AB = I_n$ and $B = A^{-1}$. To see this observe that $\ker(BA) = \ker(I_n) = \left\{\vec{0}\right\}$. As $\ker(A) \subset \ker(BA) = \left\{\vec{0}\right\}$ have $\ker(A) = \left\{\vec{0}\right\}$, and so A is invertible. Finally,

$$A^{-1} = I_n A^{-1} = (BA)A^{-1} = B(AA^{-1}) = BI_n = B$$

.