201 Linear Algebra, Practice Final Solutions

1. $z = 0$, $y = 1$, $x + y = 2$. Therefore the solutions are $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

2. The characteristic polynomial $P_A(x) = (1-x)^3$. Therefore there is one eigenvalue $\lambda = 1$ with algebraic multiplicity 3.

   The eigenspace $E_1 = \text{Ker}(A - I)$. A is diagonalizable if E_1 has dimension 3.

   $A - I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has rank 1, therefore has nullity 2. Therefore $\text{dim}E_1 = 2$. A is not diagonalizable.

   $E_1 = \text{Span}\{\vec{e}_1, \vec{e}_3\}$.

3. B is a symmetric matrix and is therefore orthogonaly diagonalizable.

   The characteristic polynomial $P_B(x) = \det \begin{pmatrix} 1-x & 2 \\ 2 & 3-x \end{pmatrix} = (1-x)(3-x) - 4 = x^2 - 4x - 1$.

   The eigenvalues are $\lambda_1 = 2 + \sqrt{5}, \lambda_2 = 2 - \sqrt{5}$.

   The eigenspaces,

   $E_{\lambda_1} = \text{Ker}(B - \lambda_1 I) = \text{Ker} \begin{pmatrix} -1-\sqrt{5} & 2 \\ 2 & 1-\sqrt{5} \end{pmatrix} = \text{Span}\{\begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix}\}$

   $E_{\lambda_2} = \text{Ker}(B - \lambda_2 I) = \text{Ker} \begin{pmatrix} -1+\sqrt{5} & 2 \\ 2 & 1+\sqrt{5} \end{pmatrix} = \text{Span}\{\begin{pmatrix} \sqrt{5}+1 \\ -2 \end{pmatrix}\}$

   The eigenspaces are perpendicular to each other since the matrix is symmetric. Let $\mathcal{B} = \{\vec{u}_1, \vec{u}_2\}$ be the orthonormal eigenbasis where

   $\vec{u_1} = \frac{1}{\sqrt{10-2\sqrt{5}}} \begin{pmatrix} \sqrt{5}-1 \\ 2 \end{pmatrix}$ and $\vec{u}_2 = \frac{1}{\sqrt{10+2\sqrt{5}}} \begin{pmatrix} \sqrt{5}+1 \\ -2 \end{pmatrix}$.

   $P = (\vec{u}_1 \vec{u}_2)$.

   The quadratic form q in $\mathcal{B}$-coordinates looks like $q(x, y) = \lambda_1 c_1^2 + \lambda_2 c_2^2$, where $\lambda_1 > 0$ ans $\lambda_2 < 0$. The level set $q(x, y) = 1$ therefore describes a hyperbola with the eigenspaces E_{λ_1} and E_{λ_2} it's principal axes.

   The point $\pm\frac{1}{\sqrt{\lambda_1}}\vec{u}_1$ are the points on the hyperbola closest to the origin.

4. $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$, $T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. The $\mathcal{S}$-matrix for T is the 4×4 matrix A such that $A \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ c \\ b \\ d \end{pmatrix}$, therefore $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

   The characteristic polynomial $P_A(x) = \det(A - xI) = (1-x)(1-x)(x^2-1) = (x-1)^3(x+1)$.

   Two eigenvalues, $\lambda_1 = 1$, algebraic multiplicity $= 3$.

   $\lambda_2 = -1$, algebraic multiplicity $= 1$.

   The eigenspaces,

$$E_1 = \text{Ker}(A-I) = \text{Ker}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\}.$$

$$E_{-1} = \text{Ker}(A+I) = \text{Ker}\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = \text{Span}\{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\}.$$

The eigenvalues of the transformation T are 1 and -1. The eigenspaces,

$E_1 = \text{Span}\{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$

$E_{-1} = \text{Span}\{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\}.$

5. Let $E_2 = \text{Span}\{v_1\}$ and $E_3 = \text{Span}\{v_2\}$. Then $\mathcal{B} = \{v_1, v_2\}$ is an eigenbasis for V with respect to T.

   The $\mathcal{B}$-matrix for $T = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$, which is invertible. Therefore T is invertible.

   The determinant of $T = \det\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = 6.$

6. Proj_V has eigenvalues 0 and 1. The eigenspaces are $E_0 = V^\perp, E_1 = V$.

   $V^\perp = \text{Ker}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} = \{\begin{pmatrix} -2t \\ -t \\ t \end{pmatrix}, t \in \mathbb{R}\} = \text{Span}\{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\}.$

   $V = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}.$

   These are bases for the eigenspaces E_0 and E_1. Notice that the eigenspaces are perpendicular to each other, therefore, we can apply Gram-Schmidt to the basis for each eigenspace asn obtain an orthonormal eigenbasis.

   Let $\{\vec{u}_1, \vec{u}_2\}$ be an orthonormal basis for $V = E_1$, and $\vec{u}_3$ an orthonormal basis for $V^\perp = E_0$. Then $Q = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$.

   (finish calculations)

7. $(1/2, \cos x, \sin x)$ is an orthonormal basis for T_1. Therefore $\text{proj}_{T_1}(x) = < x, 1/2 > 1/2 + < x, \cos x > \cos x + < x, \sin x > \sin x$.

   $< x, 1/2 > = \frac{1}{\pi}\int_{-\pi}^{\pi} x/2 dx$

   $< x, \cos x > = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos x dx$

   $< x, \sin x > = \frac{1}{\pi}\int_{-\pi}^{\pi} x \sin x dx$

   (finish calculations)

8. (a) TRUE. $\begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ has two distict eigenvalues 1 and 2, therefore is diagonalizable with the eigenvalues the diagonal entries.

(b) FALSE. $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is orthogonal.

(c) FALSE. The eigenvalues of $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$ are posititive and negetive, therefore the matrix is not positive definite.

(d) TRUE. $S^{-1}AS = D \Rightarrow S^{-1}A^2S = S^{-1}ASS^{-1}AS = D^2$ is diagonal.