201 Linear Algebra, Practice Final

Duration: 180 mins

1. Solve the system $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{x} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

2. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

   Find the eigenvalues and eigenspaces for A. Is A diagonalizable? Explain.

3. $B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$. Find an orthogonal matrix P such that $P^T B P$ is a diagonal matrix.

   Consider the quadratic form $q(x, y) = \begin{pmatrix} x & y \end{pmatrix} B \begin{pmatrix} x \\ y \end{pmatrix}$. Sketch the curve $q(x, y) = 1$. Find the point on this curve that is closest to the origin.

4. Let $T : \mathbb{R}^{2\times2} \to \mathbb{R}^{2\times2}$ be the linear transformation $T(A) = A^T$.

   Let $\mathcal{S}$ be the standard basis for $\mathbb{R}^{2\times2}$. Find the $\mathcal{S}$-matrix A for T.

   Find out the eigenvalues and the eigenspaces for the matrix A.

   Use this to find the the eigenvalues and eigenspaces for the transformation T.

5. Let V be a 2-dimensional vector space and $T : V \to V$ a linear transformation with eigenvalues 2 and 3. Show that T is an invertible transformation.

   What is the determinant of T?

6. Let $V = \text{Span}\{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$. Let A be the $\mathcal{S}$-matrix for the transformation $\text{Proj}_V : \mathbb{R}^3 \to \mathbb{R}^3$.

   Find an orthogonal matrix Q so that $Q^T A Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

7. Let C^∞ be the space of infinitely differentiable functions $[-\pi, \pi] \to \mathbb{R}$ equipped with the inner product $< f, g >= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$. Let $T_1 = \text{Span}\{1, \cos x, \sin x\}$ be the subspace of C^∞. Find the orthogonal projection of $t \in C^\infty$ on T_1.

8. TRUE or FALSE. Justify your answer.

   (a) There exists invertible matrix S such that $S^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} S = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.

   (b) The determinant of an orthogonal matrix is always positive.

   (c) $< \vec{x}, \vec{y} >= \vec{x}^T \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \vec{y}$ defines an inner product on $\mathbb{R}^2$.

   (d) A is a diagonalizable matrix implies A^2 is diagonalizable.