1. (a)

$$T(1) = 1 + 1'' = 1 = 1.1 + 0.x + 0.x^2$$
$$T(x) = x + x'' = x = 0.1 + 1.x + 0.x^2$$
$$T(x^2) = x^2 + (x^2)'' = x^2 + 2 = 2.1 + 0.x + 1.x^2$$

The $\mathcal{S}$-matrix of $T = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

(b) The change of basis matrix $S_{\mathcal{B}\to\mathcal{S}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$.

If B is the $\mathcal{B}$-matrix of T,

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

2. (a) FALSE. Similar matrices have the same determinant.
   (b) FALSE. T can be a linear transformation of rank 0 or 1.

3. $\text{Ker}(\text{Proj}_V) = V^{\perp} = \text{Ker}\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{pmatrix} = \{\begin{pmatrix} 2w - z \\ -2w \\ z \\ w \end{pmatrix} |z, w \in \mathcal{R}\}$.

$\{\begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix}\}$ is a basis.

Apply Gram-Schmidt to this basis.

$u_1 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$.

$$u_2 = \frac{\begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} - \left(\begin{pmatrix} 2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}\right)\begin{pmatrix} -1\sqrt{2} \\ 0 \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}{||"||}$$

(a) TRUE. Inverses of orthogonal matrices are orthogonal. Product of orthogonal matrices are orthogonal.

(b) FALSE. A is scaling by 2, B is scaling by 1/2. Then BA is the identity transformation.

4. The least squares solution is the solution to the system $A^T A\vec{x}^* = A^T\vec{b}$.

In this case, $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix} \vec{x}^* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$. Therefore, $\vec{x}^* = \begin{pmatrix} 1 \\ 8/5 \end{pmatrix}$.

The orthogonal projection of $\vec{b}$ on the image of A is $A\vec{x}^* = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 8/5 \end{pmatrix}$.