May 7, 2009

Name ....

Section/ Name of your TA ....

FINAL EXAM *200pts.*

MATH 201 VER ****

- **There are 12 pages in the exam excluding this page.**
- **Write all your answers clearly. You have to show work to get points for your answers.**
- **Read all the questions carefully and make sure you answer all the parts.**
- **Questions 1-8 have parts in them which are inter-related.**
- **You can write on both sides of the paper. Indicate that the answer follows on the back of the page.**
- **Use of Calculators is *not* allowed during the exam.**

(1) ..../20

(2) ..../20

(3) ..../20

(4) ..../15

(5) ..../15

(6) ..../15

(7) ..../20

(8) ..../15

(9) ..../32

(10) ..../28

Total ....../200

1 *20pts.* Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) Is A diagonalizable? explain why or why not?

2 *20pts.* Let A be a 2×2 matrix with eigenvalues $\frac{1}{2}$ and $\frac{-1}{2}$. Let

$$\text{Ker } (A - \frac{1}{2}I) = \text{Span}\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\}$$

and

$$\text{Ker } (A + \frac{1}{2}I) = \text{Span}\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}.$$

(a) Let

$$\vec{x}(t+1) = \begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = A\vec{x}(t).$$

Given that $\vec{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, find $\vec{x}(3)$.

(b) Draw the phase potrait for the discrete system in part (a).

3 *20pts.* Let $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$.

(a) Given that $\lambda = 0$ and 2 are the only eigenvalues of A. Find an orthonormal basis of $\mathbb{R}^3$ denoted by $\mathcal{B}$ consisting of eigenvectors of A.

(b) Given the following quadratic form $q(x_1, x_2) = 3x_1^2 + 4x_1x_2 + 3x_2^2$. Describe q in terms of $\mathcal{B}$ coordinates. Show work.

4 *15pts.* Let f denote a infinitely differentiable function on $\mathbb{R}$. Find all real solutions to the following differential equation.

$$\frac{d^2 f}{dt^2} - f(t) = 0.$$

5 *15pts.*

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & 0 \end{bmatrix}$$

(a) Find the inverse of A, if it exists.

(b) Give a basis of the Image of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $T(\vec{x}) = A\vec{x}$.

6 *15pts*

$$\begin{bmatrix} 1 & 1 & | & -2 \\ 1 & 2 & | & 1 \\ 1 & 1 & | & h \end{bmatrix}$$

(a) Given that the above is the augmented matrix of a system of equations, find h such that it is consistent.

(b) For $h = 0$ find the least squares solution to the system.

7 *20pts.* Let $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix} \right\}$ be two different bases of a subspace W in $\mathbb{R}^3$.

(a) Which of the two sets are orthogonal? Show work.

(b) Let $\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Is $\vec{y} \in W$?

(c) Find $\text{proj}_W \vec{y}$, that is, the orthogonal projection of $\vec{y}$ onto W.

8 *15pts.* Let A be a 2×2 matrix with eigenvalues 1 and 3, such that Ker $(A-I) =$ Span$\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}\}$ and Ker $(A-3I) =$ Span$\{\begin{bmatrix} 1 \\ -3 \end{bmatrix}\}$.

(a) Find A. Show work.

(b) Let T denote the transformation $T\vec{x} = A\vec{x}$. Write down the matrix of the transformation T with respect to the basis $\{\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \end{bmatrix}\}$. Show work.

9 *32pts.* Answer the following in short. **Give justification for your answers.**

(i) Let $\mathcal{D}$ denote the space of differentiable functions from $\mathbb{R} \to \mathbb{R}$. Is the function $<,>: \mathcal{D} \times \mathcal{D} \to \mathbb{R}$ defined as

$$< f, g >= f(0)g'(0) + f'(0)g(0)$$

an inner product on $\mathcal{D}$?

(ii) Let $V = \text{Span}\{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -2 \end{bmatrix}\}$. Find the dimension of V.
Explain your answer.

9(iii) Let A be a 2×2 matrix with eigenvalues $-1 \pm 2i$. Then consider the system of differential equations,

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

What happens to $x(t)$ as $t \to \infty$? Show work.

9(iv) Let A be an 2×2 matrix such that $A^3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then find Ker A.

10 *28pts.* State true or false with justification.

10(i) If A is a orthogonal 3×3 matrix then $\det A > 0$.

10(ii) Let $W = \text{Span}\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ and $\vec{w}_2 \in \text{Span}\{\vec{w}_1, \vec{w}_3\}$. Then $W = \text{Span}\{\vec{w}_1, \vec{w}_3\}$.

10(iii) Let $T : V \to W$ be an invertible linear transformation from a vector space V to another vector space W. If $\{v_1, v_2, v_3\}$ is a linearly independent subset of V, then $\{Tv_1, Tv_2, Tv_3\}$ is a linearly independent set in W.

10(iv) If A is a 2×2 symmetric matrix then all its eigenvalues are positive real numbers.

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