THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences

110.201 - Linear Algebra
Midterm Exam - Spring Session 2010

Instructions: This exam has 13 pages. No calculators, books or notes allowed.

• **You must answer the first 3 questions, then answer one of question 4 or 5.** **Do not answer both. No extra points will be rewarded.**

Question 6 is bonus.

- **Place an "X" through the question you are not going to answer.**
- **You must use a pen.**

You have: 50 MINUTES.

Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper: **ask the Instructor to get additional paper**. If you use extra paper, be sure to clearly label each problem and attach the extra paper to the exam.

Academic Honesty Certification

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ______________________________ Date: ____________________

Name of the student **AND session nb. (or TA's name)**: ______________________________

Problem	Score
1	
2	
3	
4 or 5	
6 (Bonus)	
Total	

1a. [10 points] Let $\mathcal{E} = \{\overrightarrow{e}_1, \overrightarrow{e}_2, \overrightarrow{e}_3\}$ be the canonical basis of $\mathbb{R}^3$. Let

$$\overrightarrow{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \ \overrightarrow{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \ \overrightarrow{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}.$$

- Prove that $\mathcal{B} = \{\overrightarrow{v}_1, \ \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a basis of $\mathbb{R}^3$.
- Write the matrix $S = S_{\mathcal{E}\to\mathcal{B}}$ that describes the change of basis from $\mathcal{E}$ to $\mathcal{B}$.

1b. [15 points] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\overrightarrow{e}_1) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ T(\overrightarrow{e}_2) = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \ T(\overrightarrow{e}_3) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

- Determine the matrix A of T (i.e. $T = T_A$) with respect to the canonical basis $\mathcal{E}$ and determine also the matrix B of T with respect to the basis $\mathcal{B}$ as in **1a.**, i.e. define explicitly $T_B : (\mathbb{R}^3, \mathcal{B}) \to (\mathbb{R}^3, \mathcal{B})$.
- Is T_B invertible?

<u>**Solution:**</u> **1a.** $\mathcal{B}$ is a basis of $\mathbb{R}^3$ if $\mathbb{R}^3 = \text{span}\{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ i.e. the three vectors are linearly independent. Let $M = [\overrightarrow{v}_1 \ \overrightarrow{v}_2 \ \overrightarrow{v}_3] = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. Then $\mathcal{B} = \{\overrightarrow{v}_1, \overrightarrow{v}_2, \overrightarrow{v}_3\}$ is a basis of $\mathbb{R}^3$ if and only if $\text{rk}(M) = 3$. This is the case since one can easily prove that $\text{rref}(M) = I_3$.
By definition $S = [[\overrightarrow{e}_1]_{\mathcal{B}}, [\overrightarrow{e}_2]_{\mathcal{B}}, [\overrightarrow{e}_3]_{\mathcal{B}}]$. We compute separately

$$\overrightarrow{e}_1 = c_1\overrightarrow{v}_1 + c_2\overrightarrow{v}_2 + c_3\overrightarrow{v}_3 \ \Leftrightarrow \ [\overrightarrow{e}_1]_{\mathcal{B}} = \frac{1}{5}\begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$$\overrightarrow{e}_2 = d_1\overrightarrow{v}_1 + d_2\overrightarrow{v}_2 + d_3\overrightarrow{v}_3 \ \Leftrightarrow \ [\overrightarrow{e}_2]_{\mathcal{B}} = \frac{1}{5}\begin{bmatrix} 4 \\ -3 \\ -2 \end{bmatrix}$$

$$\overrightarrow{e}_3 = h_1\overrightarrow{v}_1 + h_2\overrightarrow{v}_2 + h_3\overrightarrow{v}_3 \ \Leftrightarrow \ [\overrightarrow{e}_3]_{\mathcal{B}} = \frac{1}{5}\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$$

Thus $S = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}$. Alternatively, we can compute S as $S_{\mathcal{E}\to\mathcal{B}} = S^{-1}_{\mathcal{B}\to\mathcal{E}} = M^{-1}$

1b. The matrix A such that $T = T_A$ (with respect to the canonical basis) has as columns the vectors $T(\overrightarrow{e}_i)$. That is $A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Notice that $\text{rk}(A) = 2$, this means that A is not invertible.

To find B, we apply the formula $B = SAS^{-1} = \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix}^{-1} =$
$\begin{bmatrix} -\frac{2}{5} & \frac{4}{5} & -\frac{1}{5} \\ \frac{4}{5} & -\frac{3}{5} & \frac{2}{5} \\ \frac{1}{5} & -\frac{2}{5} & \frac{3}{5} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ -1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -15 & -17 & 8 \\ 15 & 19 & -11 \\ 15 & 11 & 1 \end{bmatrix}$. Since A is not invertible, B is not invertible.

2. [25 points] Given the subspace of $\mathbb{R}^{2\times 2}$ (= linear space of 2×2 real matrices)

$$S = \{\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in \mathbb{R}^{2\times 2} \;:\; \begin{bmatrix} 1 & -\frac{4}{3} \end{bmatrix} \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} 2 \\ -3 \end{bmatrix} = 0\},$$

find its dimension and a basis $\mathcal{B}$ of S.

<u>**Solution:**</u> We write out the equation defining S by performing the indicated matrix multiplication:

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in S \iff 2x - 3y + 4z = 0$$

To find the dimension of S, we observe that the equation defining S is the equation of a plane in $\mathbb{R}^3$, thus $\dim S = 2$. A basis for S is given for example by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 3/4 \end{bmatrix}.$$

In fact, there is an isomorphism of S onto the plane P in $\mathbb{R}^3$ defined by $2x - 3y + 4z = 0$, namely

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

We have $A_1 \to \begin{bmatrix} 1 \\ 0 \\ -1/2 \end{bmatrix} = \vec{v}_1$ and $A_2 \to \begin{bmatrix} 0 \\ 1 \\ 3/4 \end{bmatrix} = \vec{v}_2$ and these two vectors in P are linearly independent.

3. [25 points] Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix}$.

3a. Determine a basis of the column space of A.

3b. Determine a basis of the nullspace of A (i.e. a basis of $\mathrm{Ker}(A)$).

3c. For what value(s) of $r \in \mathbb{R}$ is the following system solvable

$$A\overrightarrow{x} = \begin{bmatrix} r \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

<u>**Solution:**</u> We have

$$\mathrm{rref}(A) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Thus a basis of the column space of A is given by the first 3 column vectors of A.

Notice first that by the rank-nullity theorem, $\dim(\mathrm{Ker}(A)) = 2$. Moreover, it is easy to see that the fourth column of A can be written as first column of A plus second column of A. From this observation we obtain $\begin{bmatrix} 1 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \in \mathrm{Ker}(A)$. In a similar way we discover that the fifth column of A is equal to the first column of A plus the third column of A. Thus we obtain a second vector in the kernel of A i.e. $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}$.

These two vectors are linearly independent and so they form a basis of $\mathrm{Ker}(A)$.

The system in **3c.** is solvable if and only if the vector $\begin{bmatrix} r \\ 0 \\ 0 \\ 1 \end{bmatrix} \in \mathrm{Image}(A)$. This is equivalent to find the solution(s) of the system

$$c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} r \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

There is a unique solution $(c_1 = 0, c_2 = -1, c_3 = 1, r = 1)$.

4. [25 points] (**ANSWER THIS QUESTION OR NUMBER 5**)

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false or motivate your answer:

(a) The matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is similar to $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(b) The matrix $B = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ defines a linear isomorphism $T_B : \mathbb{R}^3 \to \mathbb{R}^3$.

(c) If $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an orthonormal basis and $T : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear transformation, then $\{T(\vec{v}_1), T(\vec{v}_2), T(\vec{v}_3)\}$ is an orthonormal basis.

(d) If a matrix A is similar to an invertible matrix, then A is invertible.

Solution: (a) is false: if A were similar to I_2, then the following equation should hold $AS = S$, for some invertible matrix S. But that would mean that $A = I_2$.

(b) is false since by inspection we notice that $\text{Ker}(B) \supsetneq \{\vec{0}\}$ (i.e. the third column of B is a linear combination of the first 2 columns).

(c) is false since T although invertible, may not be an orthogonal transformation. Take for example $T = 2(\text{identity})$. Then $T(\vec{v}_i)$ are orthogonal vectors but not unitary.

(d) is true since if A is similar to an invertible matrix B we know that the following equation applies: $AS = SB$ for some invertible matrix S. We then get $A = SBS^{-1}$ and a product of invertible matrices is invertible.

5. [25 points] **(ANSWER THIS QUESTION OR NUMBER 4)**

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false:

(a) The matrix $M = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ has a QR factorization.

(b) If $\mathcal{A} = \{f, g\}$ and $\mathcal{B} = \{g, f - g\}$ are two bases of a linear space V, then the change of basis matrix from $\mathcal{B}$ to $\mathcal{A}$ is $S = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$.

(c) If A and B are two 2×3 matrices with $\text{Image}(A) = \mathbb{R}^2 = \text{Image}(B)$, then $\text{Ker}(A) = \text{Ker}(B)$.

(d) If a matrix A is similar to a matrix B then $\text{Ker}(A)$ is isomorphic to $\text{Ker}(B)$.

Solution: (a) is false since the QR factorization exists for matrices whose columns are made by linearly independent vectors. This is not the case for the matrix M.

(b) is false since $[f - g]_{\mathcal{A}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and this is not the second column of S.

(c) is false since $\text{Ker}(A)$ maybe different, although isomorphic, to $\text{Ker}(B)$. Take for example the case $A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Then it is easy to see that $\text{Image}(A) = \mathbb{R}^2 = \text{Image}(B)$, but $\text{Ker}(A) = \lambda \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ whereas $\text{Ker}(B) = \lambda \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ for $\lambda \in \mathbb{R}$. Thus these kernels are two isomorphic lines in $\mathbb{R}^3$ but they are not the same lines.

(d) is true since then $A = SBS^{-1}$ for some invertible matrix S, thus $\text{Ker}(A) \simeq \text{Ker}(B)$.

6. [20 points] **(BONUS: ANSWER THIS QUESTION TO GET EXTRA POINTS)**

6a. Find an orthonormal basis for the subspace $V \subset \mathbb{R}^3$ spanned by the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

6b. Find a normal (i.e. unit) basis for the orthogonal complement of V.

__**Solution:**__ **6a.** We apply the Gram-Schmidt algorithm

$$\vec{u}_1 = \frac{1}{||\vec{v}_1||}\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Then

$$\vec{v}_2^{\perp} = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Therefore $\vec{u}_2 = \sqrt{\frac{2}{3}}\begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$

6b. To find $V^{\perp} \subset \mathbb{R}^3$ we can simply take the span of $\vec{u}_1 \wedge \vec{u}_2 = \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Notice that the wedge product of two unit vectors is a unit vector. Thus $\vec{u}_3$ is the requested normal basis of $V^{\perp}$.