# THE JOHNS HOPKINS UNIVERSITY
## Faculty of Arts and Sciences

## 110.201 - Linear Algebra
## Midterm Exam - Spring Session 2010

**Instructions:** This exam has 13 pages. No calculators, books or notes allowed.

• **You must answer the first 3 questions, then answer one of question 4 or 5. Do not answer both. No extra points will be rewarded.**

**Question 6 is bonus.**

• **Place an "X" through the question you are not going to answer.**

• **You must use a pen.**

**You have: 50 MINUTES.**

Be sure to show all work for all problems. No credit will be given for answers without work shown. If you do not have enough room in the space provided you may use additional paper: **ask the Instructor to get additional paper**. If you use extra paper, be sure to clearly label each problem and attach the extra paper to the exam.

Academic Honesty Certification

I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ______________________________ Date: ____________________

Name of the student **AND session nb. (or TA's name)**: ______________________________

| Problem | Score |
|---|---|
| 1 | |
| 2 | |
| 3 | |
| 4 or 5 | |
| 6 (Bonus) | |
| **Total** | |

**1a.** [15 points] Given the following system of equations:

$$\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$$

find all solutions using Gauss-Jordan elimination procedure. Is this an example of a consistent system? Why?

**Solution:** Compute the reduced row-echelon form of the augmented matrix

$$B = \begin{bmatrix} 1 & -3 & 1 & 1 \\ 1 & 1 & 2 & 14 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & \frac{7}{4} & \frac{43}{4} \\ 0 & 1 & \frac{1}{4} & \frac{13}{4} \end{bmatrix} = \operatorname{rref}(B)$$

The solutions of the system are

$$x = \frac{43}{4} - \frac{7}{4}z, \quad y = \frac{13}{4} - \frac{1}{4}z, \quad z = \text{any real number}$$

The system has infinitely many solutions and is consistent.

**1b.** [10 points] Can the following system of equations:

$$\begin{aligned} x - y + z &= 1 \\ x + y - 2z &= 2 \\ 2x - z &= 3 \end{aligned}$$

have a single solution? Justify your answer.

**Solution:** The system has a unique solution if and only if the rank of the matrix of the coefficients is 3. We therefore compute such rank

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -2 \\ 2 & 0 & -1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(A)$$

Since $\operatorname{rk}(A) = 2 < 3$ the system cannot have a single solution.

**2.** [25 points] Given the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Can the system

$$A \cdot \overrightarrow{x} = \overrightarrow{b}, \qquad \overrightarrow{x} \in \mathbb{R}^5$$

have infinitely many solutions for some vector $\overrightarrow{b} \in \mathbb{R}^4$ and be inconsistent for some other vector $\overrightarrow{b_1} \in \mathbb{R}^4$? If yes, give two such vectors. Explain your answer.

**Hint.** Study the reduced row-echelon form of the matrix $(A \mid \overrightarrow{b})$.

**<u>Solution:</u>** A necessary condition for the system $A \cdot \overrightarrow{x} = \overrightarrow{b}$ to have infinitely many solutions, for some $\overrightarrow{b} \in \mathbb{R}^4$, is that the rank of the matrix $A$ of the coefficients is less than 5. This condition is satisfied since $\mathrm{rk}(A) \leqslant 4$ anyway.
Moreover, a necessary condition for the system $A \cdot \overrightarrow{x} = \overrightarrow{b_1}$ to be inconsistent, for some $\overrightarrow{b_1} \in \mathbb{R}^4$, is $\mathrm{rk}(A) < 4$. We therefore compute this rank

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \mathrm{rref}(A)$$

Thus $\mathrm{rk}(A) = 3$ and the question is plausible. We determine the reduced row-echelon form of the augmented matrix

$$\begin{bmatrix} A & \overrightarrow{b} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 1 & 2 & b_1 \\ -1 & 1 & 1 & 0 & 0 & b_2 \\ 0 & 1 & 1 & 1 & 1 & b_3 \\ 1 & 0 & 1 & 1 & 2 & b_4 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & -b_2 + b_3 \\ 0 & 1 & 0 & 1 & 0 & -b_1 - b_2 + 2b_3 \\ 0 & 0 & 1 & 0 & 1 & b_1 + b_2 - b_3 \\ 0 & 0 & 0 & 0 & 0 & b_4 - b_1 \end{bmatrix}$$

It follows that for $\overrightarrow{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_1 \end{bmatrix}$, with $b_1, b_2, b_3 \in \mathbb{R}$, the system has infinitely many solutions and for $\overrightarrow{b_1} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$, with $b_4 \neq b_1$, $b_4 \in \mathbb{R}$, the system is inconsistent.

**3.** [25 points] Given the following transformations:

(a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2 \quad f_1(x_1, x_2) = (x_1 + 2x_2, e^{x_1+x_2})$

(b) $f_2 : \mathbb{R}^3 \to \mathbb{R}^5 \quad f_2(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3, 3x_1 + x_2, x_1, 5x_3)$

(c) $f_3 : \mathbb{R}^3 \to \mathbb{R}^3 \quad f_3(x_1, x_2, x_3) = (x_1, x_2 + 1, x_3)$

(d) $f_4 : \mathbb{R}^3 \to \mathbb{R}^3 \quad f_4(x_1, x_2, x_3) = (x_1 + x_2, x_2, x_1 - x_3)$

- Which are linear?
- For each linear transformation in the above list write the associated matrix.
- Which of the linear transformations in the above list are invertible?
- For each invertible linear transformation in the above list write the inverse transformation explicitly.

**Solution:** $(a)$ $f_1$ is not linear as the exponential function is not a linear map.
$(b)$ $f_2$ is linear: $f_2(\overrightarrow{x}) = (f_2)_A(\overrightarrow{x})$, with

$$A = \begin{bmatrix} f_2(\overrightarrow{e_1}) & f_2(\overrightarrow{e_2}) & f_2(\overrightarrow{e_3}) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

However, $A$ is not invertible as $A$ is not a square matrix.
$(c)$ $f_3$ is not linear since $x_2 + 1$ is not homogeneous.
$(d)$ $f_4$ is a linear transformation: $f_4(\overrightarrow{x}) = (f_4)_B(\overrightarrow{x})$, with

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Moreover, $f_4$ is invertible since $\mathrm{rk}(B) = 3$. We compute the inverse by applying the Gauss-Jordan procedure

$$\begin{bmatrix} B & I_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} = \mathrm{rref}(\begin{bmatrix} B & I_3 \end{bmatrix}) = \begin{bmatrix} I_3 & B^{-1} \end{bmatrix}$$

Thus $f_4^{-1}(\overrightarrow{x}) = (x_1 - x_2, x_2, x_1 - x_2 - x_3)$.

**4.** [25 points] (**ANSWER THIS QUESTION OR NUMBER 5**)

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false:

(a) Let $A$, $B$ and $C$ be three $(2 \times 2)$-matrices.

$$AB = AC \implies B = C$$

(b) If the $3 \times 3$ matrix $A$ has rank 3, then any linear system with coefficient matrix $A$ has a unique solution.

(c) The vector $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ is orthogonal to the vector $\begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$

(d) For a $n \times m$ matrix $A$ and 2 vectors $\overrightarrow{b_1}, \overrightarrow{b_2} \in \mathbb{R}^m$ the following statement holds

$$A \cdot \overrightarrow{x} = \overrightarrow{b_i} \text{ inconsistent system, for } i = 1,2 \implies A \cdot \overrightarrow{x} = (\overrightarrow{b_1} + \overrightarrow{b_2}) \text{ inconsistent system}$$

<u>**Solution:**</u> $(a)$ is false: take $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$. Then $AB = 0 = AC$ but $B \neq C$.

$(b)$ is true since $A$ is invertible.

$(c)$ is true as $1 \cdot 0 + 0 \cdot 1 + 1 \cdot 2 + 2 \cdot (-1) = 0$

$(d)$ is false: take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$, $\overrightarrow{b_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\overrightarrow{b_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Note that $\mathrm{rk}(A) = 1$ and that $A \cdot \overrightarrow{x} = k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, as $k$ varies in $\mathbb{R}$. The system $A \cdot \overrightarrow{x} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ has infinitely many solutions: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$,...., $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$,...

**5.** [25 points] **(ANSWER THIS QUESTION OR NUMBER 4)**

State whether the following statements are true or false. If true explain your answer, if false give an example for which the statement is false:

(a) If $A$ is a $2 \times 2$ matrix such that $A^2 = A \cdot A = 0$, then $A = 0$ (the zero matrix).

(b) If $A$, $B$ and $C$ are $2 \times 2$ invertible matrices, then $(A + B)C$ is invertible.

(c) $\overrightarrow{u}$, $\overrightarrow{v}$ orthogonal, unit vectors $\Longrightarrow$ $\overrightarrow{u} + \overrightarrow{v}$, $\overrightarrow{u} - \overrightarrow{v}$ orthogonal vectors.

(d) The matrix $\begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$ describes a reflection about a line in $\mathbb{R}^2$.

**<u>Solution:</u>** $(a)$ is false: take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$(b)$ is false: take $A = \mathrm{I}_2$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $C = A = \mathrm{I}_2$ and note that $A + B$ is not invertible.

$(c)$ is true: $(\overrightarrow{u} + \overrightarrow{v}) \cdot (\overrightarrow{u} - \overrightarrow{v}) = |\overrightarrow{u}|^2 - |\overrightarrow{v}|^2 - \overrightarrow{u} \cdot \overrightarrow{v} + \overrightarrow{u} \cdot \overrightarrow{v} = 0$.

$(d)$ is false since the matrix of a reflection about a line in $\mathbb{R}^2$ is of type $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, with $a^2 + b^2 = 1$.

**6.** [20 points] **(BONUS: ANSWER THIS QUESTION TO GET EXTRA POINTS)**

Define the linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ (i.e. write the matrix $A$) satisfying the following requirements:

(a) $T_A(\overrightarrow{x}) = \overrightarrow{0}$ if (and only if) $\overrightarrow{x} = (c)\overrightarrow{b} = \begin{bmatrix}1\\0\\1\end{bmatrix}$, $(c \in \mathbb{R})$

(b) $T_A(x, y, z) = \{(x, y, z) \in \mathbb{R}^3 \mid x + z = 0\}$ for all $\begin{bmatrix}x\\y\\z\end{bmatrix} \neq \overrightarrow{b}$

Is such $T_A$ invertible? Justify your answer.

**Hint.** $\overrightarrow{b}$ is orthogonal to the plane $x+z = 0$. Thus $T_A(\overrightarrow{x}) = \overrightarrow{x} - (\overrightarrow{x})^{\perp}$, where $(\overrightarrow{x})^{\perp}$ is the orthogonal projection of $\overrightarrow{x}$ along (the line through) $\overrightarrow{b}$.

**<u>Solution:</u>** $T_A(\overrightarrow{x})$ describes the orthogonal projection of a vector $\overrightarrow{x} \in \mathbb{R}^3$ onto the plane $P$ defined by the equation $x + z = 0$.

We let $\overrightarrow{u} = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}$ be the unit vector orthogonal to the plane $P$. Then

$$T_A(\overrightarrow{x}) = \overrightarrow{x} - (\overrightarrow{x})^{\perp} = \frac{1}{2}\begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\-1 & 0 & 1\end{bmatrix} \cdot \overrightarrow{x}$$

$T_A$ is not invertible since the transformation describes a projection onto a plane and $\mathrm{rk}(A) = 2 < 3$.

<u>Remark:</u> The Hint clearly suggests to consider as $T_A$ the linear transformation orthogonal projection onto the plane $x+z = 0$. For such transformation $T_A(\overrightarrow{x}) = \overrightarrow{0}$ iff $\overrightarrow{x}$ is any vector on the line orthogonal to that plane.