April 1, 2009

Name ....

Section/ Name of your TA ....

SUGGESTED SOLUTIONS TO MIDTERM EXAM 2 *100pts.*

MATH 201 VER ****

- **There are 6 pages in the exam including this page.**
- **Write all your answers clearly. You have to show work to get points for your answers.**
- **Read all the questions carefully and make sure you answer all the parts.**
- **You can write on both sides of the paper. Indicate that the answer follows on the back of the page.**
- **Use of Calculators is *not* allowed during the exam.**

(1) ..../22

(2) ..../20

(3) ..../22

(4) ..../36

Total ....../100

(1) *22 pts.* Let $A = \begin{bmatrix} 2 & 2 & -1 \\ 1 & 0 & 0 \\ 4 & 1 & 0 \end{bmatrix}$.

(a) Find the determinant of A. Show work.

Expanding along third column

$$\det A = (-1) \cdot (1 \cdot 1 - 4 \cdot 0) = -1$$

(b) Find the classical adjoint of A. Show work

$$\operatorname{Adj} A = \begin{bmatrix} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 4 & 1 \end{vmatrix} \\ -\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 4 & 1 \end{vmatrix} \\ \begin{vmatrix} 2 & -1 \\ 0 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 0 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} \end{bmatrix}^T$$

The symbol $|A|$ denotes determinant of the matrix A.

$$= \begin{bmatrix} 0 & 0 & 1 \\ -1 & +4 & 6 \\ 0 & -1 & -2 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 & 0 \\ 0 & +4 & -1 \\ 1 & 6 & -2 \end{bmatrix}$$

(c) What is the inverse of A? Show work.

Inverse of A exists since $\det A \neq 1$

$$A^{-1} = \frac{\operatorname{Adj} A}{\det A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -4 & 1 \\ -1 & -6 & 2 \end{bmatrix}$$

(2) *20 pts.* Let $\mathcal{P}_1$ be the set of polynomials with degree less than or equal to 1, that is, $\mathcal{P}_1 = \{f(t) = a_0 + a_1 t : a_0, a_1 \in \mathbb{R}\}$. Then both $\mathcal{B}_1 = \{1, t\}$ and $\mathcal{B}_2\{1, t-1\}$ are bases of $\mathcal{P}_1$.

(a) What is the S matrix that transforms a vector in $\mathcal{B}_2$-coordinates into $\mathcal{B}_1$-coordinates. Show work.

We need to write basis elements of B_2 in terms of B_1 bases; $1 = 1.1 + 0.t$

$t-1 = -1.1 + 1.t$

Therefore, $S = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

(b) Let $T : \mathcal{P}_1 \to \mathcal{P}_1$ be the transformation defined as $T(a_0 + a_1 t) = a_0 + a_1(2t-1)$. Find the matrix B of the transformation T with respect to the basis $\mathcal{B}_2$. Show work.

The matrix of T w.r.to $\mathcal{B}_2$ $= \begin{bmatrix} [T(1)]_{B_2} & [T(t-1)]_{B_2} \end{bmatrix}$

Now. $[T(1)]_{B_2} = [1.1 + 0.(t-1)]_{B_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$[T(t-1)]_{B_2} = [-1 + 2t - 1]_{B_2} = [2t-2]_{B_2} = [0.1 + 2(t-1)]_{B_2} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

(c) Let $f \in \mathcal{P}_1$ be written as $[f]_{\mathcal{B}_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in $\mathcal{B}_2$-coordinates. Let T be as in part (b), then find $[T(f)]_{\mathcal{B}_1}$, that is, find $T(f)$ in $\mathcal{B}_1$-coordinates. Show work.

$[T(f)]_{B_1} = S\,[T(f)]_{B_2} = SB[f]_{B_2}$

$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[T(f)]_{B_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

(3) *22pts.* (a) Let $\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\}$ and $\{\begin{bmatrix}0\\1\\2\\1\end{bmatrix}, \begin{bmatrix}-2\\-1\\0\\-1\end{bmatrix}\}$ be two different sets in $\mathbb{R}^4$ spanning the same subspace. Which of these two sets are orthogonal? Show work.

$$\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix}0\\1\\2\\1\end{bmatrix}\cdot\begin{bmatrix}-2\\-1\\0\\-1\end{bmatrix} = -1-1 = -2 \neq 0$$

$$\therefore \left\{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\} \text{ is a orthogonal set.}$$

(b) Let $\{\underset{\vec{v}_1}{\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}}, \underset{\vec{v}_2}{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}}, \underset{\vec{v}_3}{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}}\}$ be a basis of a subspace V of $\mathbb{R}^4$. Find an orthonormal basis of V. Show work.

$$\vec{u}_1 = \frac{1}{\|\vec{v}_1\|}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}, \quad \text{since } \vec{v}_1 \cdot \vec{v}_2 = 0$$

$$\vec{u}_2 = \frac{1}{\|\vec{v}_2\|}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{V}_3^{\perp}}{\|V_3^{\perp}\|} = \frac{\vec{v}_3 - (\vec{v}_3\cdot\vec{u}_1)\vec{u}_1 - (\vec{v}_3\cdot\vec{u}_2)\vec{u}_2}{\|\vec{v}_3^{\perp}\|}$$

P.T.O

$$= \frac{1}{\|v_3^{\perp}\|}\left[\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - \left(\underbrace{\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}-1/\sqrt2\\0\\1/\sqrt2\\0\end{bmatrix}}\right)\frac{1}{\sqrt2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} - \left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix}\cdot\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\right)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\right]$$

$$= \frac{1}{\|v_3^{\perp}\|}\left(\begin{bmatrix}0\\0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}-1\\0\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\right)$$

$$\vec{u}_3 = \frac{1}{\|v_3^{\perp}\|}\left(\begin{bmatrix}1/4\\-1/4\\1/4\\-1/4\end{bmatrix}\right) = \frac{1}{1/2}\begin{bmatrix}1/4\\-1/4\\1/4\\-1/4\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix}$$

$$\therefore \left\{ \begin{bmatrix}-1/\sqrt2\\0\\1/\sqrt2\\0\end{bmatrix}, \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}, \begin{bmatrix}1/2\\-1/2\\1/2\\-1/2\end{bmatrix} \right\}$$ is a

orthonormal basis.

(4) *36 pts.* These are all short answer questions. Explain your answer. Each of these problems is worth 12 points.

(a) Let $\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 6$. Find $\det \begin{bmatrix} a-d & b-e & c-f \\ g & h & i \\ 4d & 4e & 4f \end{bmatrix}$. Explain your answer.

The matrix $\begin{bmatrix} a-d & b-e & c-f \\ g & h & i \\ 4d & 4e & 4f \end{bmatrix}$ is obtained

From $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ by (i) adding two rows $R1-R2$ (does not change determinant

(ii) exchanging $R2 \leftrightarrow R3$ multiplies determinant by -1

(iii) multiplying R3 by 4 multiplies det. by 4

$\therefore \det \begin{bmatrix} a-d & b-e & c-f \\ g & h & i \\ 4d & 4e & 4f \end{bmatrix} = 6 \cdot (-1) \cdot 4 = -24$

(b) The space of polynomials of degree less than or equal to 1, $\mathcal{P}_1$ is isomorphic to the space of complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$. State true or false. Give reasons.

TRUE

The space $P_1 = \{a_0 + a_1 t \mid a_0, a_1 \in \mathbb{R}\}$ is a 2 dimensional vector space with a basis $\{1, t\}$.

$\mathbb{C}$ is a 2-dimensional vector space with a basis $\{1, i\}$. and are hence isomorphic.

Alternately

Define $T: P_1 \to \mathbb{C}$ as

$a_0 + a_1 t \to a_0 + i a_1$

Then. $T(c_1(a_0 + a_1 t) + c_2(b_0 + b_1 t)) = c_1 a_0 + i c_1 a_1 + c_2 b_0 + c_2 i b_1$
$= c_1(a_0 + i a_1) + c_2(b_0 + i b_1)$
$= c_1 T(a_0 + a_1 t) + c_2 T(b_0 + b_1 t)$

Then T is ~~an~~ linear transformation

and. $T(a_0 + a_1 t) = 0 = 0 + i0$

$\Rightarrow \quad a_0 + ia_1 = 0 + i0$

$\Rightarrow \quad a_0 = a_1 = 0$

$\Rightarrow$ T is one-one.

Also for any $a + ib \in \mathbb{C}$ there is $a + bt \in P_1$ s.t $T(a+bt) = a+ib$

$\therefore$ T is onto.

Hence T is an isomorphism.

(One can also just note that the ~~linea~~ matrix of the linear transf w. respect to basis $\{1, t\}$ & $\{1, i\}$ is given by $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is invertible.)

(c) Find the volume of the parallelepiped which has the vectors $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$ as three edges. Show work and explain your answer.

Volume of a Parallelopiped formed by $\vec{v_1}, \vec{v_2}, \vec{v_3}$

$= \left| \det [\vec{v_1} \ \vec{v_2} \ \vec{v_3}] \right|$

$= \left| \det \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 2 \\ 1 & 1 & 0 \end{bmatrix} \right|$

$= | 2 \cdot (-2) + (-1)(-2) + 0 \cdot (-3) |$

$= | -4 + 2 | = 2$