

February 25, 2009

Name
Section/ Name of your TA

SUGGESTED SOLUTIONS TO MIDTERM EXAM 1 *100pts.*
MATH 201 VER*

- There are 6 pages in the exam including this page.
- Write all your answers clearly. You have to show work to get points for your answers.
- Read all the questions carefully and make sure you answer all the parts.
- You can write on both sides of the paper. Indicate that the answer follows on the back of the page.
- Use of Calculators is *not* allowed during the exam.

(1)/20

(2)/22

(3)/22

(4)/36

Total/100

(1) 20 pts. Solve the following system of equations.

$$\begin{aligned}x_1 + x_3 &= 1 \\x_1 + 2x_3 &= 0 \\x_1 + 2x_2 + 2x_3 &= 1\end{aligned}$$

Soln: The augmented matrix for the above system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 2 & 1 \end{array} \right]$$

R2- R1 and R3- R1 *implies*

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & 1 & 0 \end{array} \right]$$

Exchanging rows R2 and R3 we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

R2-R3 and R1- R3 \implies .

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 2 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array} \right]$$

Therefore, $x_1 = 2$, $x_2 = \frac{1}{2}$ and $x_3 = -1$.

(2) 22 pts. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined as

$$T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_3 \\ 0 \end{bmatrix}$$

(a) Find a basis of the Image of T .

Soln: The Image of T is the set of vectors in \mathbb{R}^3 which are of the form $T(\vec{y})$ for some $\vec{y} \in \mathbb{R}^3$.

Then

$$\begin{aligned} T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) &= \begin{bmatrix} y_1 + y_2 \\ y_1 - y_3 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} y_1 \\ y_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -y_3 \\ 0 \end{bmatrix} \\ &= y_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y_3 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \end{aligned}$$

Then Image of $T = \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right\}$

Clearly $\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right\}$ is a linearly independent set and $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$.

Therefore, the basis for image of $T = \left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right\}$.

(b) What is the Rank of the matrix defining the linear transformation T ? Explain your answer.

Soln: The matrix representing T is $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$. This will have Rank 2 since the dimension of Image $T = \text{Rank of the matrix } A$.

- (3) 22 pts. Let $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a + b + c = 0 \right\}$ (that is, the set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ satisfying the equation $a + b + c = 0$).
- (a) Show that W is a subspace of \mathbb{R}^3 .

Soln: The subspace W is the set of solutions to the matrix equation

$$[111] \begin{bmatrix} a \\ b \\ c \end{bmatrix} = [0]$$

Then we have one equation in three unknowns. We have infinitely many choices for b and c . Further

$$a = -b - c.$$

Therefore, vectors in W are of the form

$$\begin{aligned} \begin{bmatrix} a \\ b \\ c \end{bmatrix} &= \begin{bmatrix} -b - c \\ b \\ c \end{bmatrix} \\ &= \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix} \\ &= b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{aligned}$$

$$\text{Therefore, } W = \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Alternatively,

* We see that $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in W$ since $0 + 0 + 0 = 0$.

* For any two vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ and $\begin{bmatrix} e \\ f \\ g \end{bmatrix}$ in W . We know that

$$a + b + c = 0 \text{ and } e + f + g = 0. \text{ Then } \begin{bmatrix} a \\ b \\ c \end{bmatrix} + \begin{bmatrix} e \\ f \\ g \end{bmatrix} =$$

$$\begin{bmatrix} a + e \\ b + f \\ c + g \end{bmatrix}. \text{ But } a + b + c + e + f + g = 0 + 0 = 0.$$

* For any $k \in \mathbb{R}$ and $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in W$ we have that $a+b+c = 0$. Then

$$k(a+b+c) = ka+kb+kc = 0 \implies \begin{bmatrix} ka \\ kb \\ kc \end{bmatrix} \in W \text{ for all } k \in \mathbb{R}.$$

Then W is a subspace.

(b) Find a basis of W .

Soln: Clearly, no scalar multiple of $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ can be equal to $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The basis of W is $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ since they span the set and are linearly independent.

- (4) *36pts.* State whether the following statements are true or false. If true, explain your answer. If false, give an example for which the statement is false. Only TWO points for stating true or false. Each of this problem is worth 12 points.

(q) Let A be a 2×3 matrix. If the Rank $A = 2$ then the equation $A\vec{x} = \vec{b}$ has a solution for any $\vec{b} \in \mathbb{R}^2$.

Soln: This statement is true. If we have a matrix of Rank A then the $\text{rref}(A) = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$ or $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ where the $*$ positions can be non-zero. But this implies we will have two equations in three unknowns and this will always have infinitely many solutions as we will have one free variable.

(b) Let A , B and C be 2×2 matrices. Then $AB = AC$ implies $B = C$.

Soln: Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then both $AB = AC = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. But clearly, $B \neq C$.

(c) Let W be a subspace of \mathbb{R}^4 . If $W = \text{Span}\{\vec{w}_1, \dots, \vec{w}_k\}$ for vectors $\vec{w}_1, \dots, \vec{w}_k$ in \mathbb{R}^4 and the dimension of W is 3 then $k = 3$.

Soln: This statement is false. If we take a subspace of \mathbb{R}^4 ,

$$W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Then this is a subspace of \mathbb{R}^4 spanned by four vectors but the dimension is only 3.