February 25, 2009

Name ....

Section/ Name of your TA ....

SUGGESTED SOLUTIONS TO MIDTERM EXAM 1 *100pts.*
MATH 201 VER ***

- **There are 6 pages in the exam including this page.**
- **Write all your answers clearly. You have to show work to get points for your answers.**
- **Read all the questions carefully and make sure you answer all the parts.**
- **You can write on both sides of the paper. Indicate that the answer follows on the back of the page.**
- **Use of Calculators is *not* allowed during the exam.**

(1) ..../20

(2) ..../22

(3) ..../22

(4) ..../36

Total ....../100

(1) *20pts.* Solve the following system of equations.

$$\begin{aligned} x_1 + 2x_3 &= 1 \\ x_1 + 2x_2 + 2x_3 &= 0 \\ x_1 + x_3 &= 2 \end{aligned}$$

Soln: The augmented matrix for this system is

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 2 & 2 & 0 \\ 1 & 0 & 1 & 2 \end{array}\right]$$

R2 - R1 and R3- R1 $\Longrightarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

R1+ 2R1 $\Longrightarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & -1 & 1 \end{array}\right]$$

We then have that $x_1 = 3$, $x_2 = -\frac{1}{2}$ and $x_3 = -1$.

(2) *22 pts.* Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the linear transformation defined as

$$T\left(\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = \begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ y_3 \end{bmatrix}$$

(a) Find the Kernel of T.

Soln: The Kernel of T is the set of elements $\vec{y} \in \mathbb{R}^3$ such that $T(\vec{y}) = \vec{0}$.

In particular, for any $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

$$\begin{bmatrix} y_1 + y_2 \\ y_1 - y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This implies $y_3 = 0$. Moreover $y_1 + y_2 = 0$ and $y_1 - y_2 = 0 \implies 2y_1 = 0 \implies y_1 = 0$.

Hence, $y_2 = 0$. Kernel $T = \{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$

(b) Is T invertible? Explain why or why not?

Soln: Kernel of $T = \vec{0}$. This implies that if A is the matrix representing T, it has Rank $= 3$. This implies A is invertible. Therefore, T is invertible.

(3) *22 pts.* Is $W = \{\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix} : a, b \in \mathbb{R}\}$ a subspace of $\mathbb{R}^3$ (This is the set of vectors in $\mathbb{R}^3$ of the form $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix}$ for all possible real values of a and b) ?

(a) Show that W is a subspace of $\mathbb{R}^3$.

Soln: Every vector in W is of the form

$$\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Therefore, $W = \text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\}$. This implies W is a subspace. Alternatively,

* For $a = b = 0$ we have that $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.
* For $a, b, c, d \in \mathbb{R}$, we have, $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix} + \begin{bmatrix} c-d \\ d \\ 0 \end{bmatrix} = \begin{bmatrix} (a+c)-(b+d) \\ b+d \\ 0 \end{bmatrix} \in W$ for $a+c, b+d \in \mathbb{R}$.
* For any $k \in \mathbb{R}$ and $\begin{bmatrix} a-b \\ b \\ 0 \end{bmatrix} \in W$, $\begin{bmatrix} k(a-b) \\ kb \\ 0 \end{bmatrix} = \begin{bmatrix} ka-kb \\ kb \\ 0 \end{bmatrix} \in W$.

This implies that W is a subspace.

(b) Find a basis for W.

Soln: We already know that W is spanned by $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

But these two vectors are linearly independent since scalar multiples of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are of the form $\begin{bmatrix} k \\ 0 \\ 0 \end{bmatrix}$ and can never be equal to $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

Therefore a basis of W is the set $\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\}$.

(4) *36 pts.* State whether the following statements are true or false. If true, explain your answer. If false, give an example for which the statement is false. Only TWO points for stating true or false. Each of this problem is worth 12 points.

(a) Let W be a subspace of $I\!R^4$. If $W = \text{Span}\{\vec{w}_1, \cdots, \vec{w}_k\}$ for vectors $\vec{w}_1, \cdots, \vec{w}_k$ in $I\!R^4$ and the dimension of W is 3 then $k = 3$.

Soln: This statement if false. If we take a subspace of $I\!R^4$,

$$W = \text{Span}\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}\}.$$

Then this is a subspace of $I\!R^4$ spanned by Four vectors but the dimension is only 3.

(b) Let A and B be 2×2 matrices. Then $(A+B)^{-1} = A^{-1} + B^{-1}$.

Soln: This statement is false. Let $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is not even invertible even though A and B are invertible.

(c) Let A be a 2×3 matrix. If the Rank $A = 2$ then the equation $A\vec{x} = \vec{0}$ has a unique solution.

Soln: This statement is false. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Then A has rank 2 but $A\vec{x} = \vec{0}$ has infinitely many solutions since x_3 is a free variable.