Linear Algebra Review

Note: This outline is NOT intended to be exhaustive. Rather, it is intended to provide a conceptual guide to help you BEGIN to study. It is my hope that once you have thoroughly understood the material I have prepared for you here, it will be easier for you to go back and fill in the numerous gaps. Indeed, a number of topics, including all of chapter 5, were deliberately left out. Finally, please note that the review guide was made without consultation with Professor Santhanam and it is highly probable that the final will contain material outside the scope of this outline. It is also possible that this document contains typos or other mistakes. It is your responsibility to avoid being misled by such errors.

1 Row-Reduction

Row-reduction consists of the following 3 operations:
1) subtract a multiple of one row from another

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 3 & 1 \\ 2 & 9 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 3 & 3 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

2) Swap two rows.

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 3 & 1 \\ 2 & 9 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 4 & 2 \\ 2 & 9 & 6 \\ 3 & 3 & 1 \end{pmatrix}$$

3) Multiply a row by a scalar other than 0.

$$\begin{pmatrix} 1 & 4 & 2 \\ 3 & 3 & 1 \\ 2 & 9 & 6 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 8 & 4 \\ 3 & 3 & 1 \\ 2 & 9 & 6 \end{pmatrix}$$

A matrix is in **row-reduced echelon form (rref)** if and only if

- The first nonzero entry in any row is a 1, called a **leading 1**,
- If a column contains a leading 1, all other entries in that column are 0,
- If two rows each have a leading 1, then the row with the leading 1 further to the right is above the other, and

- Rows with only zeros (if any) appear at the bottom.

To solve a system of linear equations, we may enter the coefficients and constant into a matrix and find the row-reduced echelon form of that matrix. For example:

$$\begin{array}{rcrcl} 2x & + & y & = & 8 \\ -x & + & 2y & = & 1 \end{array} \rightarrow \begin{pmatrix} 2 & 1 & 8 \\ -1 & 2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix}$$

In this case, $x = 3$ and $y = 2$.
It is also possible to have infinitely many solutions. You should practice finding the solution sets to a few such examples from the text.
It is also possible to have no solution. Make sure you can identify when this happens.

The **rank** of a matrix A is the number of rows of rref(A) that are not all zero. A system of equations has a unique solution if and only if the rank of the corresponding matrix is equal to the number of variables.

2 Real Vector Spaces, Subspaces, and Linear Transformations

A **real vector space** is a set of objects (called vectors) such that for any vectors x, y, and z and any real numbers a and b:

- $(x + y) + z = x + (y + z)$
- $x + y = y + x$
- There is a vector called 0 with the property that $x + 0 = x$ for any x
- For each vector x there is a vector y so that $x + y = 0$ (Such a vector is called $-x$)
- $a(x + y) = ax + ay$
- $(a + b)x = ax + bx$
- $a(bx) = (ab)x$
- $1x = x$

Examples of real vector spaces include $\mathbb{R}^n$, $\mathbb{C}$, the space P_n of polynomials with degree $\leq n$, the space of (continuous) functions from $\mathbb{R}$ to $\mathbb{R}$, and the space of infinite sequences, to name a few. Your book uses the term "**linear space**" instead of real vector space.

A **subspace** is a set of vectors inside a larger vector space so that:

- If x is in the subspace, so is ax (Therefore 0 is in the subspace)
- If x and y are in the subspace, so is $x + y$

A set of vectors x_j is linearly independent if and only if
whenever $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$, $a_1 = a_2 = \cdots = a_n = 0$. In other words, no vector in the set can be written as a linear combination of the others.
A set of n vectors in $\mathbb{R}^n$ is linearly independent if and only if the vectors are the columns (or the rows) of a rank n matrix.
The **span** of a set of vectors is the let of all linear combinations of those vectors.
The **dimension** of a vector space is the number of linearly independent vectors needed to span the space. For example, it takes n linearly independent vectors to span $\mathbb{R}^n$. Conversely, if V is an n-dimensional vector space, *any* set of n linearly independent vectors will span V.

A linear transformation $T : V \to W$ is a map from one vector space (V) to another (W) so that:

- $T(x + y) = T(x) + T(y)$
- $T(ax) = aT(x)$

An $n \times m$ matrix is a linear transformation from $\mathbb{R}^m$ to $\mathbb{R}^n$

The **kernel** of a linear transformation $T : V \to W$ is the set of vectors x in V so that $T(x) = 0$. The kernel of T is a subspace of V.

The **image** of a linear transformation $T : V \to W$ is the set of vectors y in W so that $T(x) = y$ for some x in V. In other words, the image is the set of vectors that T maps *onto*. If we represent T as a matrix, the image of T is the span of its columns. For this reason, some books use the word **column space** instead of image.

Two vector spaces V and W are **isomorphic** if there is an one-to-one and onto linear transformation (called an **isomorphism**) between them.

T is an isomorphism if and only if the following three conditions hold:

- $\ker(T) = \{0\}$
- $\mathrm{Im}(T) = W$
- $\dim(V) = \dim(W)$

If any two of the above conditions hold, so does the third.

The **Rank Nullity Theorem**: For a linear transformation $T : V \to W$,
$\dim(V) = \dim(\mathrm{Im}(T)) + \dim(\ker(T))$

3 Matrix Algebra and Determinants

A square matrix is **invertible** if and only if the corresponding linear transformation is an isomorphism.
An $n \times n$ matrix is invertible if and only if its rank is n
An $n \times n$ matrix is invertible if and only if its kernel is $\{0\}$
To find the inverse of A, row-reduce the matrix $(A|I_n)$. The result should be I_n in place of A, and A^{-1} in place of I_n. For example:

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ -2 & -5 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 3 \\ -2 & -5 \end{pmatrix}^{-1} = \begin{pmatrix} -5 & -3 \\ 2 & 1 \end{pmatrix}$$

If A is an $n \times m$ matrix and B is a $p \times q$ matrix, we can multiply $A \times B$ if $m = p$. Their product AB is an $n \times q$ matrix.
Matrix multiplication is associative ($A(BC) = (AB)C$) and distributive ($A(B + C) = AB + AC$, $(A + B)C = AC + BC$), but NOT commutative (in general, $AB \neq BA$).
For a square matrix A, $(A)(A^{-1}) = I_n$

The **determinant** of a square matrix is the factor by which the matrix expands areas.

$$\det \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

.

$$\det \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} \qquad = aei + bfg + cdh - gec - ahf - dbi$$

For a matrix larger than 3×3, the easiest way to find the determinant is to row-reduce the matrix to a triangular matrix. An **upper-triangular** matrix is a matrix with only 0s below the main diagonal. A **lower-triangular** matrix is a matrix with only 0s above the main diagonal. The determinant of a triangular matrix is the product of the entries on the main diagonal. Row-reduction has the following effect on the determinant:

- Multiplying a row by k multiplies the determinant by k
- Swapping two rows multiplies the determinant by -1
- Adding a multiple of one row to another has no effect on the determinant

det(AB)=det(A)det(B). If A is $n \times n$, det(A)=0 if and only if rank(A)¡n. The **transpose** of A is given by replacing the first row of A with the first column, the second row with the second column, and so on. For example

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

$\det(A)$=$\det(A^T)$
$\det(A^{-1})$=$\frac{1}{\det(A)}$

The **adjoint** of A is $\frac{1}{\det(A)}A^{-1}$

Two matricies A and B are called **similar** if there is an invertible matrix S so that $AS = SB$. Similar matricies have the same determinant.

4 Coordinates, Eigenvalues, Eigenvectors and Diagonalization

Given any basis for a vector space, it is possible to represent each vector in that space uniquely as a linear combination of basis elements. For example, consider the basis

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

If we want to represent the vector $\begin{pmatrix} 4 \\ -5 \end{pmatrix}$ with respect to that basis, we note that

$$\begin{pmatrix} 4 \\ -5 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix} + 9\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

so in our new coordinates, we write the vector as $\begin{pmatrix} 4 \\ 9 \end{pmatrix}$. To convert *back to standard coordinates*, we apply the transformation

$$S = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$$

$$S\begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$

Since S is the matrix that takes us from the new coordinates back to standard coordinates, the linear transformation from standard coordinates into the new coordinates is S^{-1}.

Given the matrix A in standard coordinates of a linear transformation T, if we wish to find a matrix B that applies the same transformation to vectors written in the new basis, we note that the transformation $B = S^{-1}AS$ converts from new coordinates to standard, then applies T, then converts back to new coordinates, which is exactly what we want. Note that A and B are similar matricies.
Another useful way to find the transformation given by A in new coordinates is to see how A acts on the coordinate vectors of the new coordinate system. For example, if

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$\text{Then } B = \begin{pmatrix} 3 & 0 \\ 2 & 1 \end{pmatrix}$$

because $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ in new coordinates, and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in new coordinates.

If A is a square matrix, an **eigenvector** of A is a vector that is mapped onto a multiple of itself. If $Ax{=}\lambda x$ for a scalar λ, we say λ is the **eigenvalue** associated with x.
To find the eigenvalues of a square matrix A, Set the determinant of $A - \lambda I$ equal to 0 and solve for λ. For example, let $A = \begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix}$

$$\det \begin{vmatrix} 3-\lambda & -2 \\ -1 & 4-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

The above equation is called the **characteristic equation** of A, and its roots, $\lambda = 2$ and $\lambda = 5$, are the eigenvalues of A. The **algebraic multiplicity** of λ is the number of times λ appears as the root of the characteristic equation.

The **eigenspace** of an eigenvalue λ is the set of vectors x that map to λx. (Note: an eigenspace is associated to an *eigenvalue*. It makes NO sense to talk about the eigenspace of a *matrix*, because the eigenvectors of the matrix do not, in general, form a subspace!) To find the eigenspace associated with λ, find the kernel of $A - \lambda I$. For $\lambda = 2$,

$$\text{eigenspace}(2) = \ker \begin{pmatrix} 3-2 & -2 \\ -1 & 4-2 \end{pmatrix} = \ker \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} = \text{span} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $\lambda = 5$,

$$\text{eigenspace}(2) = \ker \begin{pmatrix} 3-5 & -2 \\ -1 & 4-5 \end{pmatrix} = \ker \begin{pmatrix} -2 & -2 \\ -1 & -1 \end{pmatrix} = \text{span} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Eigenvalues can also be complex numbers.

The **geometric multiplicity** of an eigenvalue is the dimension of its associated eigenspace. It is always at least one, but never greater than the algebraic multiplicity of the eigenvalue.

A matrix is called **diagonalizable** if it is similar to a diagonal matrix. A matrix is diagonalizable if and only if the sum of the geometric multiplicities of the eigenvalues add up to the dimension of the space.

In our example, we are working in $\mathbb{R}^2$. There are two eigenvalues, each with geometric multiplicity one, so A is diagonalizable. Using eigenvectors as coordinates, we can write the transformation represented by A as a diagonal matrix with eigenvalues on the diagonal. So $A = S^{-1}DS$.

$$\begin{pmatrix} 3 & -2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}$$