1. (a) Suppose $\vec{0} = a_1\vec{x}_1 + a_2\vec{x}_2$. Applying $(A-\lambda_1 I)$ to both sides gives $\vec{0} = (A-\lambda_1 I)(a_1\vec{x}_1 + a_2\vec{x}_2) = a_1(A-\lambda_1 I)\vec{x}_1 + a_2(A-\lambda_1 I)\vec{x}_2 = a_2(\lambda_2-\lambda_1)\vec{x}_2$. Hence $a_2 = 0$ because $(\lambda_2 - \lambda_1)\vec{x}_2 \neq \vec{0}$, and the original equation now implies that $a_1 = 0$: the vectors are linearly independent.

(b) Suppose $\vec{0} = a\vec{x} + b\vec{y} + c\vec{z}$. Taking the dot product with $\vec{x}$, we get $0 = a\vec{x}\cdot\vec{x} + b\vec{x}\cdot\vec{y} + c\vec{x}\cdot\vec{z} = a\vec{x}\cdot\vec{x}$. Hence $a = 0$ because $\vec{x} \neq 0$. In the same way, taking the dot product with $\vec{y}$, resp. $\vec{z}$, yields $b = 0$, resp. $c = 0$. Thus the vectors are linearly independent.

2. (a) We simply compute: $P\vec{y}\cdot(\vec{x}-P\vec{x}) = \vec{y}\cdot P^\top(\vec{x}-P\vec{x}) = \vec{y}\cdot P(\vec{x}-P\vec{x})$ because $P^\top = P$, and $\vec{y}\cdot P(\vec{x}-P\vec{x}) = \vec{y}\cdot(P\vec{x}-P^2\vec{x}) = 0$ because $P = P^2$.

(b) The assumption is that $\dim(\ker P) = 0$. The image of P is therefore the whole space $\mathbf{R}^n$, because by the rank-nullity theorem the rank of P is $n - \dim(\ker P) = n$. Thus every vector $\vec{z} \in \mathbf{R}^n$ is of the form $P\vec{y}$ (for some vector $\vec{y}$), and so part (a) implies that $\vec{z}\cdot(\vec{x}-P\vec{x}) = 0$ for every vector $\vec{z}$. Thus $\vec{x} - P\vec{x} = \vec{0}$. But because $\vec{x}$ is an arbitrary vector, this means that P is the identity matrix.

3. (a) The two columns of A are not scalar multiplies of each other, so they are linearly independent. Thus for a basis of $\operatorname{Im} A$ we can simply take the columns of A, namely

$$\vec{x}_1 = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix}0\\1\\1\end{bmatrix}.$$

From these vectors the Gram-Schmidt procedure gives the following orthonormal basis of $\operatorname{Im} A$:

$$\vec{u}_1 = \frac{\vec{x}_1}{\|\vec{x}_1\|} = \begin{bmatrix}1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix}, \quad \vec{u}_2 = \frac{\vec{x}_2 - (\vec{x}_2\cdot\vec{u}_1)\vec{u}_1}{\|x_2 - (\vec{x}_2\cdot\vec{u}_1)\vec{u}_1\|} = \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}.$$

(b) The QR decomposition of A is

$$\begin{bmatrix}1 & 0\\1 & 1\\0 & 1\end{bmatrix} = A = QR = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{6}\\1/\sqrt{2} & 1/\sqrt{6}\\0 & 2/\sqrt{6}\end{bmatrix}\begin{bmatrix}\alpha_1 & \vec{x}_2\cdot\vec{u}_1\\0 & \alpha_2\end{bmatrix},$$

where $\alpha_1 = \|\vec{x}_1\| = \sqrt{2}$, $\alpha_2 = \|x_2 - (\vec{x}_2\cdot\vec{u}_1)\vec{u}_1\| = \sqrt{3/2}$, and $\vec{x}_2\cdot\vec{u}_1 = 1/\sqrt{2}$.

(c) Recall that a least squares solution $\vec{x}$ satisfies the condition $A^\top(A\vec{x}-\vec{b}) = \vec{0}$ (since this is equivalent to the condition that $A\vec{x}$ is the orthogonal projection of $\vec{b}$ onto $\operatorname{Im} A$). The matrix $A^\top A$ is invertible because its kernel equals $\ker A = \{\vec{0}\}$. Hence there is unique a least squares solution, which is $\vec{x} = (A^\top A)^{-1}A^\top\vec{b}$. From the QR decomposition and its properties ($Q^\top Q = I$, R is invertible) we have $(A^\top A)^{-1}A^\top = R^{-1}Q^\top$, and so the least squares solution is

$$\vec{x} = R^{-1}Q^\top\vec{b} = \begin{bmatrix}1/\sqrt{2} & -1/\sqrt{6}\\0 & \sqrt{2/3}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2} & 0\\-1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\end{bmatrix}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2/3\\2/3\end{bmatrix}.$$

4. (a) The trace and determinant are $\operatorname{Tr} A = 2$, $\det A = -3$. The characteristic polynomial is $\det(A-\lambda I) = \lambda^2 - (\operatorname{Tr} A)\lambda + \det A = \lambda^2 - 2\lambda - 3 = (\lambda-3)(\lambda+1)$.

(b) Since the matrix is symmetric there is an orthonormal basis of eigenvectors. From the characteristic polynomial we see that the eigenvalues are 3 and -1. By inspection we see that $[1,1]^\top$ is a 3-eigenvalue. The (-1)-eigenvalue must be orthogonal to this, so by inspection, again, we see that $[1,-1]^\top$ is a (-1)-eigenvector. Thus when we normalize these eigenvectors and put them into the columns of a matrix to form the othogonal matrix

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix},$$

we get $U^\top AU = \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix}$.

(c) Express $[1,0]^\top$ as a linear combination of the eigenvectors, and then apply A^4: since

$$\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\left(\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}1\\-1\end{bmatrix}\right),$$

we get, by linearity,

$$A^4\begin{bmatrix}1\\0\end{bmatrix} = \frac{1}{2}\left(A^4\begin{bmatrix}1\\1\end{bmatrix} + A^4\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{2}\left(3^4\begin{bmatrix}1\\1\end{bmatrix} + (-1)^4\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}41\\40\end{bmatrix}.$$