TEST 1 SOLUTIONS

1a. Supposing that $0 = a_1 T(v_1) + a_2 T(v_2) + a_3 T(v_3)$, we have to show that $a_1 = a_2 = a_3 = 0$, necessarily.

We have: $0 = a_1 T(v_1) + a_2 T(v_2) + a_3 T(v_3)$

$$\underset{\text{because } T \text{ is linear}}{=} T(a_1 v_1 + a_2 v_2 + a_3 v_3)$$

Since $\ker(T) = \{0\}$, we must have $a_1 v_1 + a_2 v_2 + a_3 v_3 = 0$, and so $a_1 = a_2 = a_3 = 0$ since v_1, v_2, v_3 are lin. indep.

1b. $x + y + 2z = 0$ is the matrix eq'n.

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Since (space of sol'ns) $= \ker(A)$, the dim. of the space of sol'ns is 2. Since A is in reduced row-echelon form, we can just write down its kernel as the span of two linearly independent vectors.

$$\ker(A) = \left\{ \begin{bmatrix} -s-2t \\ s \\ t \end{bmatrix} : s,t \in \mathbb{R} \right\} = \operatorname{span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

this is a <u>basis</u>: these vectors span $\ker(A)$, and they are linearly independent (they aren't scalar multiples of each other.)

1c. No: the sum of two solutions is <u>not</u> a sol'n. (failure of closure under addition). Also, the zero vector fails to be a sol'n.

1d. Example: projection onto xy-plane, i.e., the linear transformation $P : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$P\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

A is in reduced row-echelon form, so it is evident that $\dim(\ker P) = \dim(\ker A) = 1$ and that $\dim(\operatorname{Im} P) = \dim(\operatorname{Im} A) = 2$, as required.

By the nullity-rank theorem, any lin. transf. $T : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $3 = \dim(\ker T) + \dim(\operatorname{Im} T)$, so $\ker T$ and $\operatorname{Im} T$ cannot both be 1-dimensional.

2a. 4 rows (# rows of R.H.S.)
3 columns (# rows of matrix that A multiplies on the left)

2b.

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = A\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = A\left(2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$

$$= 2A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 3A\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

mult. by A defines a <u>linear</u> transf.

$$= 2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ 0 \\ 5 \\ 2 \end{bmatrix}.$$

3a. A is already in reduced row-echelon form, so we can immediately write down the sol'ns of $A\vec{x}=\vec{0}$:

$$\ker(A) = \left\{ \begin{bmatrix} -2a+c \\ a \\ b-3c \\ b \\ -2c \\ c \end{bmatrix} : a,b,c \in \mathbb{R} \right\} = \text{span}\,\underbrace{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right\}}_{\vec{x}\quad\vec{y}\quad\vec{z}}$$

these are linearly independent, and ∴ form a basis of ker(A), because if

$$\vec{0} = a\vec{x} + b\vec{y} + c\vec{z} = \begin{bmatrix} -2a+c \\ a \\ b-3c \\ b \\ -2c \\ c \end{bmatrix}, \text{ then evidently } a=b=c=0$$

3b. If B is row-equivalent to A, then the linear relations among the columns of B are precisely the same linear relations among the columns of A. Therefore

$$-\frac{1}{2}\vec{b_2} - 3\vec{b_4} + 2\vec{b_5} - \vec{b_6} = \vec{0}.$$

4a. By applying a few row operations, we see that

$$\left[A \,\middle|\, \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right] \sim \left[\underbrace{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}_{\text{rref}(A)} \,\middle|\, \underbrace{\begin{matrix} 1 & 2 & -2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{matrix}}_{A^{-1}} \right]$$

$\text{rref}(A) = I_3$ and so A is invertible, and its inverse is as indicated.

4b. $A\vec{x}=\vec{0} \Rightarrow \vec{x} = A^{-1}\vec{0} = \vec{0}$; $\quad A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \vec{x} = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

4c. Yes, the columns of A are linearly independent. One way to see this: A is invertible

$$\vec{0} = a_1(\text{col. 1 of } A) + a_2(\text{col. 2 of } A) + a_3(\text{col. 3 of } A)$$
$$= A\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = A^{-1}\vec{0} = \vec{0}.$$

Another way: $\text{rref}(A) = I_3$; the columns of I_3 are evidently lin. indep., and therefore so must the columns of A, because the linear relations among the columns of rref(A) are precisely the same as the linear relations among the colums of A (since A and rref(A) are row-equivalent).

5a. We have to show that $T(f+g) = T(f) + T(g)$ and that $T(af) = aT(f)$, for every $f, g \in V$ and every $a \in \mathbb{R}$. To verify this we just compute:

$$T(f+g) = (f+g) + (f+g)' = f+g+f'+g' = T(f)+T(g)$$
$$T(af) = af + (af)' = af + af' = a(f+f') = aT(f).$$

5b. $\dim V = 3$, so we only need to show that either the vectors of $\mathcal{B}$ are linearly indep. or that $\text{span}\,\mathcal{B} = V$. Let's show that the vectors of $\mathcal{B}$ are linearly indep.:

$$0 = a\cdot 1 + b\cdot(1+x) + c(1+x+x^2) = (a+b+c)\cdot 1 + (b+c)x + cx^2$$

because $1, x, x^2$ are lin. indep.

$$\iff a+b+c = b+c = c = 0$$
$$\iff a=b=c=0.$$

If $f(x) = 2 - x + 3x^2 = a + b(1+x) + c(1+x+x^2)$
then : $a+b+c = 2, \quad b+c = -1, \quad c = 3,$
i.e., $a = 3, \quad b = -4, \quad c = 3.$

5c. To find the matrix M that satisfies $[T(f)]_{\mathcal{B}} = M[f]_{\mathcal{D}}$ we have to compute $T(\vec{x})$ for each $\vec{x} \in \mathcal{D}$ and express the result in terms of the basis $\mathcal{B}$.
We have:

$$T(-x) = (-x) + (-x)' = -x - 1 = -1 \cdot (1+x)$$

$$T(x^2) = x^2 + (x^2)' = x^2 + 2x$$
$$= -2 \cdot 1 + 1 \cdot (1+x) + 1 \cdot (1+x+x^2)$$

$$T(3) = 3 + 3' = 3 + 0 = 3 \cdot 1$$

Thus, the matrix M is $\begin{bmatrix} 0 & -2 & 3 \\ -1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

5d. T is an isomorphism if and only if any matrix representing T, in particular, M, is invertible.

M is, indeed, inventible, as one can easily check by showing that $\ker(M) = \{\vec{0}\}$:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = M \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -2b+3c \\ -a+b \\ b \end{bmatrix} \Rightarrow b = 0, a = 0, c = 0.$$

Thus T is an isomorphism.