Math 110.201: Linear algebra (Spring 2008)

STUDY SHEET FOR THE FINAL EXAM

Scope of the exam

Chapters 1–8 (minus §7.6), plus the additional topics covered in lecture. The final exam is a comprehensive exam covering the content of the entire course.

Some additional practice problems

In addition to the problems from the midterm review sheets, try the following problems to get additional hands-on practice; any difficulties that you may have with them will help pinpoint gaps in your understanding/knowledge. An answer key is not provided because understanding how to check the correctness of your solution is in itself a very good gauge of your grasp of the material. If you are really stuck, ask a friend, or visit your TA or the instructor during office hours.

Be sure that you know how to do all previous homework and test problems that you either couldn't do, or hadn't done correctly. I also suggest going through some of the true/false problems at the chapter ends.

1. If A is a 3×4 matrix such that

$$A \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix},$$

then what must the numbers a, b, c be? (Are the numbers even constrained by these equations?) How large could the dimension of the kernel of A be?

2. Find a 3×3 matrix A whose 2-eigenspace is spanned by $[1,1,1]^\top$, $[1,0,1]^\top$ and whose 3-eigenspace is spanned by $[2,1,0]^\top$.

3. Compute the determinant of the following matrices:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 3 & 0 & 0 \\ 1 & 2 & 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & -A \\ I & -I \end{bmatrix}, \quad \begin{bmatrix} A & -A \\ I & -I \end{bmatrix}.$$

4. Let $\vec{x} = [-1,1]^\top$, $\vec{y} = [2,1]^\top$. If possible, find a 2×2 matrix A that satisfies the following equalities:

$$\vec{x} \cdot A\vec{x} = 6, \quad \vec{x} \cdot A\vec{y} = 0, \quad \vec{y} \cdot A\vec{x} = -6, \quad \vec{y} \cdot A\vec{y} = 3.$$

(Something to think about: Can you compute $\vec{e}_1 \cdot A\vec{e}_2$, for example? How is this number related to A?)

5. Find the least square solution to the inconsistent system of equations

$$x = 1, \quad x = 2, \quad x = 3.$$

6. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Show that the rank of A is 2 and that the dimension of $\ker A$ is 1. Find an orthogonal basis $\vec{u}$, $\vec{v}$, $\vec{w}$ of $\mathbf{R}^3$ such that $\vec{w}$ spans $\ker A$, and the vectors $A\vec{u}$, $A\vec{v}$ form an orthogonal basis of $\operatorname{Im} A$. (Hint: Think about the aim of singular value decomposition.)

7. Determine the singular value decomposition of the following two matrices:

$$\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}, \qquad \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} [1\ 2\ 1].$$

8. Let V be the set of 2×2 matrices of trace 0.

(a) Show that V is a vector subspace of the 4-dimension vector space of all 2×2 matrices. Show that it has dimension 3 by showing that the matrices

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

are linearly independent and that they span V.

(b) Let $T: V \to V$ be the linear transformation $T(A) = HA - AH$. Determine the matrix of T with respect to the basis H, X, Y. Is T an isomorphism? Justify your answer!

(c) Show that the pairing $\langle A, B \rangle = \mathrm{Tr}(A^\top B)$ is an inner product on V. Compute the inner products between the matrices H, X, Y. What are the lengths of these matrices? What is the angle between H and $X - Y$?

9. Suppose A is a 4×4 matrix whose diagonal entries are 1, -1, 3, 2. Suppose further that -1, 1, 5 are eigenvalues of A. Compute the trace of A. Since A is of size 4×4 it has one more eigenvalue: determine this eigenvalue and compute the determinant of A. Compute the determinant and trace of A^2. Is A diagonalizable? (As always, justify your answer!)

10. Which vector in the image of the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 0 \\ 1 & 2 & -1 \\ 4 & 2 & -2 \end{bmatrix}$$

is closest to the vector $[3, -2, 1, 2]^\top$? What is the minimum distance?

11. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ -1 & -3 & 2 & 5 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & -1 & -3 \end{bmatrix} \quad \text{and the vectors} \quad \vec{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{y} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ -2 \end{bmatrix}.$$

Show that $\vec{x}$ is in the image of A, and that $\vec{y}$ is in the kernel of A. Find a basis for the image of A such that $\vec{x}$ is one of the basis vectors. Find a basis for the kernel of A such that $\vec{y}$ is one of the basis vectors.

12. Solve the following system of equations via row reduction, and check your answer:

$$\begin{array}{rcrcrcrcrl} 2u & + & 4v & + & 2w & + & 4x & + & 2y & = 4 \\ & & & & w & - & x & - & y & = 4 \\ 3u & + & 6v & + & 6w & + & 3x & + & 6y & = 6 \\ 2u & + & 4v & + & 3w & + & 3x & + & 3y & = 4 \end{array}$$

Find an invertible matrix E such that $EA = \mathrm{rref}\, A$, where A is the matrix of coefficients of this system of equations. Determine the dimensions of the image and the kernel of A. Does Cramer's rule apply? Explain!

13. Let V be the linear space of polynomials of degree ≤ 2. Let $T: V \to V$ be function $T\big(p(x)\big) = xp'(x) + p(x)$, where $p'(x)$ is the derivative of $p(x)$.

(a) Determine the matrix M of T with respect to the (ordered) basis $\mathcal{B} = \{1, x, x^2\}$ of V.

(b) Show that $\mathcal{B}' = \{-x^2, 1+x, 1-x\}$ is also a basis of V. Determine the matrix N of T with respect to the basis $\mathcal{B}$ for the domain and the basis $\mathcal{B}'$ for the codomain; that is, determine the matrix N that satisfies $N[p]_{\mathcal{B}} = [T(p)]_{\mathcal{B}'}$, where, for example, $[p]_{\mathcal{B}}$ is the column vector that represents p in the basis $\mathcal{B}$.

(c) Find a matrix A such that $N = AM$. How many such matrices are there? Explain!

14. Suppose A is a 7×4 matrix with linearly independent columns. Explicitly determine the following: (a) the kernel of A; (b) the dimension of the kernel of $A^\top$; (c) the dimension of the image of $A^\top$; (d) the dimension of the image of A; (e) all solutions $\vec{x}$ of the equation $A\vec{x} = (2^{\text{nd}}$ column of $A)$; (f) the reduced row-echelon form of A.

15. For the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

determine the eigenvalues and eigenspaces. Is A positive definite? Does A have a QR decomposition? If so, compute it. Compute

$$A^{100} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

in an efficient way.

16. Suppose A is an $n \times n$ matrix such that $A^2 = I$ and that $A \neq \pm I$, where I is the identity matrix. Show that the only eigenvalues of A are -1 and 1. (In particular, show that *non-zero* eigenvectors for these eigenvalues actually exist.) If the trace of A is 0 could n be 5?

17. Suppose A is an $n \times n$ matrix such that $A^2 = -I$, where I is the identity matrix. Compute $\det(-I)$; in particular, show that n must be an even number. Give an explict example of such a matrix for $n = 2$, and for $n = 4$.

18. Consider the quadratic form $Q(x, y) = 3x^2 - 2\sqrt{2}xy + 2y^2$. Find a symmetric matrix A such that $Q(x, y) = \vec{X}^\top A \vec{X}$, where $\vec{X} = [x, y]^\top$. Find numbers a, b, c, d (not all zero!) with the following property: for all x and y, we have $Q(ax + by, cx + dy) = 4(ax + by)^2 + (cx + dy)^2$. (Hint: Diagonalize A.)

19. Let V be the linear space of polynomials of degree ≤ 2 endowed with the inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx$. Apply the Gram-Schmidt procedure to find an orthonormal basis of V from the (ordered) basis 1, $1 + x$, $1 + x + x^2$. Using this orthornormal basis, determine the polynomial $q(x) \in V$ that minimizes the integral $\int_0^1 |q(x) - \cos(\pi x)|^2\,dx$.

20. Consider the vectors

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ -2 \\ 5 \\ -4 \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} 3 \\ 2 \\ -1 \\ 0 \end{bmatrix}.$$

Among the four subspaces $\operatorname{span}\{\vec{x}_2, \vec{x}_3, \vec{x}_4\}$, $\operatorname{span}\{\vec{x}_1, \vec{x}_3, \vec{x}_4\}$, $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_4\}$, $\operatorname{span}\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$, determine which ones are the same. Compute the dimension of each of these subspaces. (Hint: Row reduce the matrix whose columns are made up of these vectors.)

21. Let $\vec{z} = [1, 1, -1, -2]^\top$ and let $T\colon \mathbf{R}^4 \to \mathbf{R}$ be the linear transformation $T(\vec{x}) = \vec{z} \cdot \vec{x}$. Show that the vector $\vec{u} = [1, 1, 0, 1]$ is in the kernel of T. Find a basis of $\ker T$ that contains $\vec{u}$. (What is the dimension of $\ker T$?)

22. For the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

show that its rank is 2, and show that $A^\top A$ is invertible without computing the product. True or false?: $A(A^\top A)^{-1} A^\top [3, 0, 7]^\top = [3, 0, 7]^\top$. (Hint: This is easy if you understand what the LHS computes.)