Math 110.201: Linear algebra (Spring 2008)

STUDY SHEET FOR TEST 2

Scope of the test

Chapters 5–8 (minus §7.6), plus the additional topics covered in lecture.

Some practice problems

Try the following problems to get additional hands-on practice; any difficulties that you may have with them will help pinpoint gaps in your understanding/knowledge. (These problems should not, however, be regarded as sample test problems, although problems of this sort may appear on the test.) An answer key is not provided because understanding how to check the correctness of your solution is in itself a very good gauge of your grasp of the material.

In addition to trying the problems here, be sure that you know how to do all previous homework problems that you either couldn't do, or hadn't done correctly. I would also strongly suggest going through some of the true/false problems at the chapter ends.

1. Compute the (orthogonal) projection of the vector $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ onto the image of the matrix

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 3 \\ 1 & 4 & 6 \end{bmatrix}.$$

2. If P is an orthogonal projection matrix (i.e., $P\mathbf{x}$ is the orthogonal projection of $\mathbf{x}$ onto the image of P), show that $P = P^2 = P^\top$. (Hint: To show that $P = P^\top$ consider the two dot products $P\mathbf{x} \cdot \mathbf{y}$ and $\mathbf{x} \cdot P\mathbf{y}$.) Determine the relation between $\det P$ and the rank of P. How is the trace of P related to the rank of P?

Conversely, if P is matrix with the property that $P = P^2 = P^\top$, must P be an orthogonal projection matrix? (Hint: What is the value of the dot product $P\mathbf{x} \cdot (\mathbf{x} - P\mathbf{x})$?)

3. Let A be a matrix. Determine the relation between $\ker(A^\top A)$ and $\ker A$, and the relation between $\operatorname{Im}(A^\top A)$ and $\operatorname{Im} A^\top$.

4. Find the volume of the parallelepiped spanned by the vectors $\mathbf{i}$, $\mathbf{i} - \mathbf{k}$, $\mathbf{i} - \mathbf{j}$.

5. Give an example of a 2×2 real matrix one of whose eigenvalues is the complex number $1 - 2i$. What must the trace and determinant of this matrix be?

6. Diagonalize the matrix

$$A = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and compute A^4.

7. Let A be a 2×2 matrix whose determinant is 3 and whose trace is -4. Is A diagonalizable? Justify your answer!

8. Suppose A is a 4×4 matrix whose complete set of eigenvalues are 1, 2, 3. Is A invertible? What is the smallest possible value for $\det A$? What is the largest possible value? Must A be diagonalizable? Justify you answers!

9. For the system of equations

$$\begin{aligned} x &+ 2y - z = 3 \\ 3x &+ y - 2z = 0 \\ -2x & + 3z = 7 \end{aligned}$$

express y as a suitable multiple of a 3×3 determinant. (Hint: Cramer's rule.)

10. Determine the quadratic polynomial $p(x)$ that minimizes the integral $\int_0^1 |p(x) - \cos(2x)|^2 \, dx$.

11. Determine the (differentiable) functions $f(x)$ and $g(x)$ that satisfy the following conditions:

$$f(0) = 1, \quad g(0) = 2, \quad f'(x) = 9f(x) + 2g(x), \quad g'(x) = 2f(x) + 6g(x).$$

12. For the matrix

$$A = \begin{bmatrix} 9 & 2 \\ 2 & 6 \end{bmatrix}$$

find a matrix B such that $B^2 = A$ (that is, find a square root of A).

13. For the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 7 & 1 \\ 3 & 2 & 5 & 9 & 1 \\ 5 & 1 & 8 & 8 & 0 \\ 4 & 0 & 8 & 2 & 3 \end{bmatrix}$$

compute $\det(A^\top A)$.

14. Determine the QR decomposition of the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Use the QR decomposition of A to solve the system of equations

$$A\mathbf{x} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 2 \end{bmatrix}$$

and to find the least square solution for the vector

$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

(that is, find the vector $\mathbf{y}$ such that $\|A\mathbf{y} - \mathbf{b}\|$ is as small as possible).

15. Compute the determinants of the matrices

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 2 & 4 \\ 0 & 0 & 3 & 1 \\ 1 & 3 & 0 & 0 \\ 4 & 2 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

16. Compute the singular value decomposition of the matrices

$$\begin{bmatrix} 6 & 3 \\ -1 & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix};$$

that is, find a factorization $UDV^\top$ (2×2 factors), where U and V are orthogonal matrices and D is a diagonal matrix whose diagonal entries are the singular values.