Practice Problems for Test 1

Linear algebra, Spring 2008
Math 201

1) Define the following basic concepts:
- linear independence } of a linear space
- basis } of a linear space
- span of vectors
- linear transformation (between linear spaces)
- kernel and image of a matrix / linear transformation
- inverse of a matrix / linear transf.
- dimension (of a linear space)

a number
- matrix of a linear transformation (depends on a choice of basis!)
- reduced row-echelon form of a matrix.
- subspace of a linear space.
- linear space
- zero vector (of a linear space)

these are numbers
- rank of a matrix / linear transformation
- nullity of a matrix / linear transformation

2) Suppose A, B, C are matrices. What must the sizes of A, B, C be so that the product ABC is defined?

How would one show that $(AB)C = A(BC)$? (There is an easy way, and there is a tedious way.)

3) Give examples of matrices A, B of size 2×2 such that $AB \neq BA$. Can you find a 7×7 example?

4) Find all solutions of the equations

$$\begin{cases} 7x_1 + 3x_2 + x_4 - x_5 = 0 \\ x_4 + x_5 = 0 \\ 2x_3 - 4x_4 + 2x_5 = 0 \end{cases}$$

5) Consider a system of 5 homogeneous equations in 13 unknowns.

True / false / not enough info?: There exists a nontrivial solution.
(remember that setting all the variables to 0 is the trivial solution)

6) Is the map $T: \mathbb{R}^3 \to \mathbb{R}^3$, $T(\vec{x}) = \vec{x} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ a linear transformation? Justify your answer.

7) Consider a system of 9 equations in 7 unknowns. How would you write this set of eq'ns in matrix form? What must be true in order that the set of solutions (assuming that solutions exist!) forms a linear space? If the set of solutions forms a linear space, what is the largest that its dimension could be?

8) Let $R: \mathbb{R}^2 \to \mathbb{R}^2$ be the (mirror) reflection about the line spanned by $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

a) Let $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Show that $\{\vec{v}_1, \vec{v}_2\}$ forms a basis of $\mathbb{R}^2$. Find the matrix of R with resp. to the basis $\{\vec{v}_1, \vec{v}_2\}$.

b) Let $\vec{v}_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Show that $\{\vec{v}_1, \vec{v}_2\}$ forms a basis of $\mathbb{R}^2$. Find the matrix of R with resp. to the basis $\{\vec{v}_1, \vec{v}_2\}$.

c) Let A = matrix in part (a)
B = matrix in part (b)
Find an invertible matrix C such that

$$CA = BC$$

9) Find the reduced row-echelon form of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}.$$

Can A be a product of elementary matrices? Justify your answer!
(this will be easy to settle once you know $\operatorname{rref}(A)$.)

10) a) Show that the (linear) transformation

$$R : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$
$$R(\vec{x}) = A\vec{x}, \text{ where } A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

expresses counterclockwise rotation in $\mathbb{R}^2$ by $\frac{\pi}{4}$ (about the origin).

b) Express A as a product of elementary matrices. (There is more than one way!)

c) Such a factorization expresses A as a succession of geometric planar transformations — explain which ones precisely for the particular factorization you found in (b).

11) Determine whether the vectors

$$\begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (\text{of } \mathbb{R}^4)$$

are linearly independent.

12) Find a basis for the subspace of $\mathbb{R}^3$ spanned by the 4 vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}.$$

What is the dimension of this subspace?

13) Find the <u>kernel</u> and <u>rank</u> of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

Find a basis for the image of A.

14) If A is a matrix such that $\ker(A) = \{\vec{0}\}$, show that $A\vec{x}_1, A\vec{x}_2, \ldots, A\vec{x}_n$ are linear independent whenever $\vec{x}_1, \ldots, \vec{x}_n$ are lin. indep.

15) Let A be the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

Find a matrix E such that

$$EA = \mathrm{rref}(A).$$

16) Suppose A and B are matrices such that

$$AB = BA.$$ (Thus A and B are square matrices — why?)

Let $\vec{x}$ be a column of B. Show that $A\vec{x}$ is in the image of B.
Show that if $\vec{x} \in \ker(B)$ then $A\vec{x} \in \ker(B)$.

17) Let T be a linear transformation such that $\ker(T) = \{0\}$. Show that $T(x) = T(y)$ can only hold if $x = y$.

18) Find a 3×5 matrix A such that $\dim(\ker A) = 2$. What is $\dim(\mathrm{Im}\, A)$?

19) Is there a matrix B such that

$$B \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 5 & 1 \\ 3 & 6 & 1 \end{bmatrix}?$$ Justify your answer.

20) Let V be the linear space of polynomials of degree ≤ 2.

Let $D: V \to V$ be the map

$$D(f(x)) = f'(x) \quad \text{(derivative)}$$

a) Determine $\ker(D)$ and $\operatorname{Im}(D)$.

b) Let $\mathcal{B}$ be the basis $\{x^2, 1, 2x\}$. Determine the <u>matrix</u> of D with respect to $\mathcal{B}$. Do the same for the basis $\mathcal{B}' = \{6, x, 3x^2\}$.

c) Let A, A' be the matrices in part (b) (A = matrix of D w.r.t. $\mathcal{B}$). Find a matrix C such that $CAC^{-1} = A'$.

c) Show that $\{1-x, 1+x, 1+x+x^2\}$ is a basis of V.

Determine the matrix of D with respect to this basis.

21) Let P be the linear space of all polynomials. Determine the dimension of

$$\operatorname{span}\{x^5 + x^2 + x,\ x^5 - x^2 + x,\ 3x^5 + x^2 + 3x,\ x^5 + x\}$$

22) Let W be as in (21). Show that the dimension of the kernel of any linear transf. $T: W \to \mathbb{R}$ is infinite.

23) a) Let V be as in (20). Find a basis for V with respect to which the matrix of $T: V \to V$

$$T(f(x)) = f(x) + f'(x)$$

is $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

b) Why is T invertible?

c) What is $T^{-1}(1 - x + x^2)$?

24) Let V_1 be the linear space of 2×2 matrices. (Quick ques.: What is $\dim(V_1)$?)
Let V_2 be the linear space of polynomials of degree ≤ 4.
Let $T: V_2 \to V_2$ be the lin. transf.

$$T(f(x)) = \int_0^1 f(x)\,dx.$$

Show that V_1 and $\ker(T)$ are isomorphic linear spaces.

25) Let V be the linear space of all 2×2 matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that $a+d=0$.

a) Show that V is indeed a linear space.

b) Let $x = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Show that $\{x, y, z\}$ is a <u>basis</u> of V.

c) Find numbers α, β such that

$$xy - yx = \alpha x,$$
$$xz - zx = \beta z.$$

(<u>easy</u>?)

d) Show that the map $A: V \to V$

$$A(u) = xu - ux$$

is a linear transformation.

e) Determine the <u>matrix</u> of A with respect to the basis $\{x, y, z\}$.
Find $\ker(A)$ and the rank of A.

26) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$. Find a basis of $\ker(A)$ that includes the vector $\begin{bmatrix} 4 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.