**Johns Hopkins University**
**Math 201, Spring 2007**
**Name:**
**Section:**

## Midterm Exam # 2

Time: 50 minutes

No books, notes, calculators. Please explain carefully all steps leading to your solutions, or risk losing credit.

**Problem 1:** (6 points=2+1+1+2) Consider the plane $E$ in $\mathbb{R}^3$ with equation $x_1 + 2x_2 + x_3 = 0$, and let $p$ denote the orthogonal projection onto $E$.

1. If $(u_1, u_2)$ is an orthonormal basis of $E$, write the formula for $p(v)$ in terms of $u_1$ and $u_2$ (where $v$ is any vector in $\mathbb{R}^3$).
2. Find a basis of $E$.
3. Find an orthonormal basis of $E$.
4. Find the matrix for $p$ in the standard basis of $\mathbb{R}^3$.

1) $p(v) = (v.u_1)u_1 + (v.u_2)u_2$

2) Take any two vectors in $E$ which are not proportional, such as: $v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$

3) We find an orthonormal basis of $E$ by applying the Gram-Schmidt procedure to $(v_1, v_2)$. This gives:

$$u_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad v_2^\perp = v_2 - (v_2 . u_1)u_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

$(u_1, u_2)$ is then an orthonormal basis of $E$.

4) We use questions 1) and 3) to find $p(e_1), p(e_2), p(e_3)$ where $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are the vectors of the standard basis in $\mathbb{R}^3$:

$$p(e_1) = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ -1/3 \\ -1/6 \end{pmatrix}, \quad p(e_2) = 0 - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ 1/3 \\ -1/3 \end{pmatrix}$$

$$p(e_3) = -\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/6 \\ -1/3 \\ 5/6 \end{pmatrix}.$$

The matrix for $p$ in the basis $(e_1, e_2, e_3)$ is therefore:

$$\begin{pmatrix} 5/6 & -1/3 & -1/6 \\ -1/3 & 1/3 & -1/3 \\ -1/6 & -1/3 & 5/6 \end{pmatrix}$$

**Problem 2:** (9 points=3+2+1+1+2) Consider the linear space $P$ of polynomials with real coefficients.

1. Are the following subsets of $P$ linear subspaces? Explain why.
   -the set $E_0$ of polynomials $p$ such that $p(0) = 0$
   -the set $E_1$ of polynomials $p$ such that $p(1) = 1$
   -the set $P_2$ of polynomials of degree 2 or less
   Consider the linear map $f$ from $P_2$ to $P_2$ defined by $f(p(x)) = p''(x) + 3p'(x)$.
2. Find the kernel of $f$. Is $f$ an isomorphism?
3. Find the matrix for $f$ in the standard basis $(1, x, x^2)$ of $P_2$.
4. Prove that the vectors $p_1 = 2 + x$, $p_2 = 3$, $p_3 = 1 + 2x + 3x^2$ are linearly independant.
5. Find the matrix for $f$ in the basis $(p_1, p_2, p_3)$.

1) * $E_0$ is a linear subspace of $P$:
- the 0 polynomial is in $E_0$
- $E_0$ is closed under addition and scalar multiplication
(if $p_1, p_2 \in E_0$, then $(p_1+p_2)(0) = p_1(0) + p_2(0) = 0$; if $\lambda \in \mathbb{R}$, $\lambda p_1(0) = \lambda \cdot 0 = 0$)

* $E_1$ is not a linear subspace of $P$: each condition fails.
the 0 polynomial is not in $E_1$, $E_1$ is not closed under addition or scalar multiplication)

* $P_2$ is a linear subspace of $P$ (seen in class and in the book): contains 0, closed under addition and scalar multiplication.

2) Let $p(x) = ax^2 + bx + c$ be in $P_2$. Then: $p'(x) = 2ax + b$, $p''(x) = 2a$

Thus: $f(p(x)) = 0 \Leftrightarrow 2a + 3(2ax + b) = 0$ (for all $x$) $\Leftrightarrow a = 0$ and $b = 0$

Therefore: $\text{Ker } f = \left\{ \begin{matrix} \text{constant} \\ \text{polynomials} \end{matrix} \right\} = \text{Span}(1)$. $f$ is not an isomorphism.

3) We find the images of the basis vectors:

$f(1) = 0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, $f(x) = 3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ $f(x^2) = 6x + 2 = \begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$

So the matrix for $f$ in $(1, x, x^2)$ is: $\begin{pmatrix} 0 & 3 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$

4) These vectors (polynomials) are linearly independant because they have different degrees. More explicitly, if $c_1p_1 + c_2p_2 + c_3p_3 = 0$:

$$c_1(2+x) + 3c_2 + c_3(1+2x+3x^2) = 0 \quad \text{for all } x$$

$$2c_1 + 3c_2 + c_3 + (c_1 + 2c_3)x + 3c_3x^2 = 0 \quad \text{for all } x$$

$$\Updownarrow$$

$$2c_1 + 3c_2 + c_3 = 0 \text{ and } c_1 + 2c_3 = 0 \text{ and } 3c_3 = 0$$

$$\Updownarrow$$

$c_1 = c_2 = c_3 = 0$. Therefore the 3 vectors are linearly independant.

5) We find the images $f(p_1), f(p_2), f(p_3)$ and write them in terms of $p_1, p_2, p_3$ (so we write the matrix column by column).

$$f(p_1) = 3 = p_2 \quad , f(p_2) = 0 \quad , f(p_3) = 18x + 12 = 18(x+2) - 24$$
$$= 18p_1 - 8p_2$$

so the matrix for $f$ in $(p_1, p_2, p_3)$ is $\begin{pmatrix} 0 & 0 & 18 \\ 1 & 0 & -8 \\ 0 & 0 & 0 \end{pmatrix}$

**Problem 3:** (5 points=1+2+2)
Consider the matrix:

$$M = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

and let $v_1$,$v_2$,$v_3$ denote its column vectors.

1. Prove that $v_1$,$v_2$,$v_3$ are linearly independant.
2. Perform the Gram-Schmidt process on $(v_1, v_2, v_3)$.
3. Write the QR-factorization of $M$.

1) If $v_3$ was a linear combination $c_1 v_1 + c_2 v_2$, then we would have: $c_1 = 1$, $c_2 = -1$, and $c_1 + 2c_2 = 1$ which is impossible.

2) $$u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{v_2^\perp}{\|v_2^\perp\|}, \quad v_2^\perp = v_2 - (v_2 \cdot u_1) u_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} - \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_2 = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$u_3 = \frac{v_3^\perp}{\|v_3^\perp\|}, \quad v_3^\perp = v_3 - (v_3 \cdot u_1) u_1 - (v_3 \cdot u_2) u_2$$

$$= \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{2}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$u_3 = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix}$$

3) Thus $M = QR$, with
$$Q = \begin{array}{c} \begin{array}{ccc} \quad u_1 \quad & \quad u_2 \quad & \quad u_3 \quad \end{array} \\ \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & -2/\sqrt{6} \\ 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \end{pmatrix} \end{array}$$

and $$R = \begin{pmatrix} \|v_1\| & v_2 \cdot u_1 & v_3 \cdot u_1 \\ 0 & \|v_2^\perp\| & v_3 \cdot u_2 \\ 0 & 0 & \|v_3^\perp\| \end{pmatrix} = \begin{pmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} & -1/\sqrt{3} \\ 0 & 0 & \sqrt{6}/3 \end{pmatrix}$$