Name: ______________________________ April 11, 2006

Section Number/Time ____________________

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MATH 201, MIDTERM #2 SOLUTIONS

Problems 1-4 will involve the linear space $V = \left\{ 2 \times 2 \text{ matrices } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$.

1. [6 points] Find a basis for V.

The elements $\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ form a basis of V.

2. [2 points] What is the dimension of V?

There are four elements in the basis, so the dimension of V is 4.

3. Given any 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we can "rotate" the entries to form a new matrix $T(A) = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$.

Find a matrix A which solves $T(A) = -A$.

We need to solve the equation $\begin{bmatrix} c & a \\ d & b \end{bmatrix} = \begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. Any scalar multiple of $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ will suffice.

4. Suppose we know that $T^4(A) = A$ for every 2×2 matrix A.
Explain why every eigenvalue of T must satisfy $\lambda^4 = 1$.

If λ is an eigenvalue, we can find a nonzero matrix A so that $T(A) = \lambda A$.
It follows that $T^2(A) = \lambda^2(A)$, $T^3(A) = \lambda^3 A$, and finally $A = T^4(A) = \lambda^4 A$.
Since A is nonzero, that forces $\lambda^4 = 1$.

In Problems 5 and 6, we consider the linear transformation $Tf(x) = xf(x) - 3\int_0^x f(t)\,dt$,
acting on the space $\mathcal{P}_2 = \{\text{All quadratic polynomials } f(x) = ax^2 + bx + c\}$.

5. If g represents the function $g(x) = 1$ for all x, what is the function Tg?

$$Tg(x) = x - 3\int_0^x 1\,dt = x - 3x = -2x.$$

6. Show that if f is any function in $\mathcal{P}_2$, then the resulting function Tf is also in $\mathcal{P}_2$.

Let $f(x) = ax^2 + bx + c$. Then
$$\begin{aligned} Tf(x) &= x(ax^2 + bx + c) - 3\left(\int_0^x at^2 + bt + c\,dt\right) \\ &= \left(ax^3 + bx^2 + cx\right) - \left(ax^3 + \tfrac{3}{2}bx^2 + 3cx\right) = -\tfrac{1}{2}bx^2 - 2cx. \end{aligned}$$

Which is again an element in $\mathcal{P}_2$.

7. What is the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix}$?

Since the matrix is upper-triangular, you can multiply the entries along the diagonal.
$\det(A) = 1 \cdot 2 \cdot 5 \cdot 10 = 100.$

8. Now consider $B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 4 & 6 \\ 0 & 0 & 5 & 7 \\ 0 & 0 & 0 & 10 \end{bmatrix}$. How is the determinant of B related to the determinant of A? *Hint:* Use row operations to turn matrix B into matrix A.

Subtracting $3 \times$ (row I) from row II does not change the determinant.
Multiplying row II by -1 multiplies the determinant by -1.
This transforms matrix B into matrix A, so $\det(B) = -\det(A)$.

9. Now consider $C = \begin{bmatrix} 0 & 0 & 0 & 10 \\ 0 & 0 & 5 & 7 \\ 0 & 2 & 5 & 6 \\ 1 & 2 & 3 & 4 \end{bmatrix}$. How is the determinant of C related to the determinant of A?

Matrix C can be turned into matrix A by swapping rows I and IV, then swapping rows II and III.
Therefore $\det(C) = (-1)^2 \det(A)$.

10. What are the eigenvalues of the matrix $M = \begin{bmatrix} -1 & -3 \\ 4 & 6 \end{bmatrix}$? Call them λ_1 and λ_2.

The determinant of $M - \lambda I$ is $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$.
The eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 3$.

11. Write down a matrix whose kernel tells you the eigenvectors of M associated to the eigenvalue λ_1.

The eigenspace associated to λ_1 is the kernel of the matrix $M - \lambda_1 I$.
Depending on whether you have chosen $\lambda_1 = 2$ or $\lambda_1 = 3$,
that matrix will be either $\begin{bmatrix} -3 & -3 \\ 4 & 4 \end{bmatrix}$ or $\begin{bmatrix} -4 & -3 \\ 4 & 3 \end{bmatrix}$.

12. Find an eigenvector $\vec{v}_1$ of M with eigenvalue λ_1.

Depending on whether you have chosen $\lambda_1 = 2$ or $\lambda_1 = 3$, $\vec{v}_1$ should be a multiple of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} 3 \\ -4 \end{bmatrix}$.

13. How large are the entries of the matrix M^{50}?

Since M can be diagonalized, we can write $M = S \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} S^{-1}$ for some invertible matrix S.
That makes $M^{50} = S \begin{bmatrix} 2^{50} & 0 \\ 0 & 3^{50} \end{bmatrix} S^{-1}$. The entries of S and S^{-1} are unremarkable.
By far the largest quantity here is 3^{50}, which is about a 24-digit number. Every entry of M^{50} will be approximately this large, either as a positive or negative number.

14. What are the eigenvalues of the matrix $A = 2 \begin{bmatrix} \cos 72° & -\sin 72° \\ \sin 72° & \cos 72° \end{bmatrix}$?

This is a rotation-scaling matrix, with eigenvalues $\lambda_{1,2} = 2(\cos(72°) \pm i \sin 72°)$.

15. Fill in the blanks: $A^5 = \begin{bmatrix} \_\_\_ & \_\_\_ \\ \_\_\_ & \_\_\_ \end{bmatrix}$.

The matrix A^5 will rotate an angle of $72° \times 5 = 360°$ and scale by a factor of $2^5 = 32$.
This makes $A^5 = 32 \begin{bmatrix} \cos 360° & -\sin 360° \\ \sin 360° & \cos 360° \end{bmatrix} = \begin{bmatrix} 32 & 0 \\ 0 & 32 \end{bmatrix}$.