Name: ______________________________ February 28, 2006

Section Number/Time ______________________

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MATH 201, MIDTERM #1 SOLUTIONS

Problems 1-11 examine a wide range of properties of the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -2 & 2 & 2 \\ 3 & -1 & 3 \end{bmatrix}$.

1. The domain of A is $\mathbb{R}^3$.

2. The codomain of A is $\mathbb{R}^4$. Other answers which describe the image of A in more detail are also acceptable.

3. To find rref(A), follow the calculations:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ -2 & 2 & 2 \\ 3 & -1 & 3 \end{bmatrix} \begin{matrix} \\ \\ +2\,\mathrm{I} \\ -3\,\mathrm{I} \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \\ 0 & -1 & -3 \end{bmatrix} \begin{matrix} \\ \\ -2\,\mathrm{II} \\ +\,\mathrm{II} \end{matrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

4. The rank of A is 2.

5. The image of A has dimension 2, same as the rank.

6. The first two columns of A, which are $\begin{bmatrix} 1 \\ 0 \\ -2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 2 \\ -1 \end{bmatrix}$ form a basis for the image.

7. The kernel of A has dimension 1, since there is column of rref(A) without a leading 1.

8. The relation among the columns is $2\vec{v}_1 + 3\vec{v}_2 - \vec{v}_3 = \vec{0}$, so any multiple of $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$ is a basis for the kernel.

9. The kernel of A contains more than just the zero vector (because x_3 is a free variable in this system of equations), so $A\vec{x} = \vec{b}$ will have infinitely many solutions whenever it has any solutions at all.

10. The bottom two rows of rref(A) contain only zeros. The vector $\vec{b}$ can be chosen to make one or both of these rows inconsistent. Therefore it is possible for $A\vec{x} = \vec{b}$ to have no solutions.

In other words, the image of A is not all of $\mathbb{R}^4$.

Problems 11-15 use the set of vectors $\mathcal{B} = (\vec{e}_1, \vec{e}_2, \vec{v}) = \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \right)$ in $\mathbb{R}^3$.

11. $\mathcal{B}$ is a basis provided the vector $\vec{v}$ is linearly independent from $\vec{e}_1$ and $\vec{e}_2$. This will be satisfied whenever the third co-ordinate v_3 is not zero.

12. The fact that $T(\vec{e}_1) = \vec{0}$ tells us that the first column of the $\mathcal{B}$-matrix will have all zeros.

The fact that $T(\vec{e}_2)$ tells us the same will be true of the second column as well.

The fact that $T(\vec{v}) = 0\vec{e}_1 + 0\vec{e}_2 + \vec{v}$ tells us that the third column of the $\mathcal{B}$-matrix should have entries of 0, 0, and 1.

So the full matrix for T according to this basis is $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

13. The image of T is span($\vec{v}$), which is the line L in the direction $\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$.

14. T is the projection onto the line L and parallel to the xy-plane. For any vector $\vec{x}$ in $\mathbb{R}^3$, $T(\vec{x})$ is the point on L at the same height (x_3 coordinate) as $\vec{x}$.

15. The matrix for T in standard coordinates can be found using the formula $A = SBS^{-1}$.

16. [2 points Extra Credit] We know that $T(\vec{e}_1) = \vec{0}$ and $T(\vec{e}_2) = \vec{0}$, and can calculate the fact that $T(\vec{e}_3) = \begin{bmatrix} \frac{v_1}{v_3} \\ \frac{v_2}{v_3} \\ 1 \end{bmatrix}$ in standard coordinates. Therefore $A = \begin{bmatrix} 0 & 0 & \frac{v_1}{v_3} \\ 0 & 0 & \frac{v_2}{v_3} \\ 0 & 0 & 1 \end{bmatrix}$.