110.201 Linear Algebra
6th Quiz

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Problem 1 Let

$$A = \begin{bmatrix} 1 & 4 \\ 1 & -2 \end{bmatrix}.$$

1. Find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$.
2. Find a formula for the entries of A^n, where n is a positive integer.

Solution The eigenvalues are 2 and -3, and we may choose $S = \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix}$, with $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$. Next,

$$A^n = S^{-1}D^nS = \begin{bmatrix} 1/5 & 1/5 \\ 1/5 & -4/5 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & (-3)^n \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 4*2^n/5 + (-3)^n/5 & 2^n/5 - (-3)^n/5 \\ 4*2^n/5 - 4*(-3)^n/5 & 2^n/5 + 4 + (-3)^n/5 \end{bmatrix}.$$

Problem 2 Let

$$A = \begin{bmatrix} 0 & -i \\ 2i & 1 \end{bmatrix}, \qquad i^2 = -1.$$

1. Find the eigenvalues of A.

2 Find A^{30}.

Solution The eigenvalues are 2 and -1. We may take $S = \begin{bmatrix} -i & i \\ 2 & 1 \end{bmatrix}$. To compute A^{30},

$$A^{30} = S^{-1}D^{30}S = \begin{bmatrix} i/3 & 1/3 \\ -2i/3 & 1/3 \end{bmatrix} \begin{bmatrix} 2^{30} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -i & i \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2^{30}/3 + 2/3 & 1/3 - 2^{30}/3 \\ 2/3 - 2*2^{30}/3 & 2*2^{30}/3 + 1/3 \end{bmatrix}.$$

Problem 3 Let A to be a 3×3 matrix with eigenvalues $1, -1, 2$.
Let $B = A^3 - 5A^2$.

1. Find the eigenvalues of B. Is B diagonalizable? If yes, find a diagonal matrix D such that $D = S^{-1}BS$, for some orthogonal matrix S.

2. Find $\det(B)$ and $\det(A - 5I_3)$.

Solution $B\vec{v} = A^3\vec{v} - 5A^2\vec{v} = (\lambda^3 - 5\lambda^2)\vec{v}$, so that plugging in the eigenvalues of A, we get -4,-6,-12 as the eigenvalues of B. Since they are distinct, B is diagonalizable, with determinant $-4 * -6 * -12 = -288$. Since $\det(A - \lambda I) = (\lambda + 4)(\lambda + 6)(\lambda + 12)$,we plug in 5 for λ to get $\det B = 9 * 11 * 17 = 1683$.