110.201 Linear Algebra
6th Quiz

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Problem 1 Let
$$A = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}.$$

1. Find an invertible matrix S and a diagonal matrix D such that $S^{-1}AS = D$.
2. Find a formula for the entries of A^n, where n is a positive integer.

Solution $\lambda_1 = 2$ and $\lambda_2 = 3$. $S = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ and $A = S^{-1}DS$. Next,
$$A^n = S^{-1}D^nS = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2^n & 0 \\ 2^n - 3^n & 3^n \end{bmatrix}.$$

Problem 2 Let
$$A = \begin{bmatrix} 1 & 2i \\ -i & 0 \end{bmatrix}, \qquad i^2 = -1.$$

1. Find the eigenvalues of A.

2 Find A^{50}.

Solution The eigenvalues are 2 and -1. One choice for S is $S = \begin{bmatrix} 2i & -i \\ 1 & 1 \end{bmatrix}$. Then
$$A^{50} = S^{-1}D^{50}S = \begin{bmatrix} -i/3 & 1/3 \\ i/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2^{50} & 0 \\ 0 & (-1) \end{bmatrix} \begin{bmatrix} 2i & -i \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2^{51}/3 - 1/3 & -2^{50}/3 - 1/3 \\ -2^{51} - 2/3 & 2^{50}/3 - 2/3 \end{bmatrix}.$$

Problem 3 Let A to be a 3×3 matrix with eigenvalues $2, -2, 1$.
Let $B = A^3 - 3A^2$.

1. Find the eigenvalues of B. Is B diagonalizable? If yes, find a diagonal matrix D such that $D = S^{-1}BS$, for some orthogonal matrix S.

2. Find $\det(B)$ and $\det(A - 3I_3)$.

Solution $B\vec{v} = A^3\vec{v} - 3A^2\vec{v} = (\lambda^3 - 3\lambda^2)\vec{v}$, so plugging in 2, -2, and 1 we get -4, -20, and -2 as the eigenvalues of B. Then B is diagonalizable since it has 3 distinct eigenvalues. We may take $D = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. The determinant of B is the product of the eigenvalues, or -160. Next, we have $\det(A - \lambda I) = (\lambda - 2)(\lambda + 2)(\lambda - 1)$. Taking $\lambda = 3$, we get 10.