Quiz 5 Solutions

April 24, 2005

Thursday

(1) This was a homework problem (HW 9). Let d_n be the determinant of the $n \times n$ matrix shown. Then by Laplace expanding along the first row, we see that

$$d_n = (-1)^{n+1} d_{n-1}$$

for $n > 1$. Using this recursion, we find that

$$d_n = (-1)^{\sum_{j=2}^{n} j+1}$$

The summation in the exponent is equal to $n(n+3)/2$. This is an even number when $n(n+3)/2$ is odd and an odd number otherwise. Elementary arithmetic shows that $n(n+3)$ is divisible by four whenever n or $n+3$ is, and not divisible by four when n and $n+3$ are not. Therefore d_n is equal to one whenever n is congruent to 1 or 4 (mod 4) and -1 otherwise.

(2) (a) Recalling that $\det AB = (\det A)(\det B)$ for $n \times n$ matrices A and B, we see that $A^2 = A$ implies that $(\det A)^2 = (\det A)$ Thus $(\det A)$ can be equal to zero or one.

(b) If $\det A = 0$ then A is not invertible, implying $m < n$. If $\det A = 1$, A is invertible and $m = n$. The equation satisfied by A is the equation obeyed by a projection operator, i.e. A represents a projection onto some subspace V of $\mathbb{R}^n$. Therefore A is equal to the identity on image(A). It now follows easily that if $m = n$, A is the identity matrix.

(3) The flaw is that $\det(-A) = (-1)^n \det A$, which can be seen by using multilinearity of the determinant function. Thus if n is even, $\det(-A) = \det A$ and we get a trivial equality $\det AB = \det BA = \det A \det B$.