Friday

(1) Use the row operations

1. (IV - III)
2. (III - II)
3. (II - I)

to reduce A to the matrix

$$A' = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

Since adding a row to another row does not change the determinant, $\det A = \det A' = 105$.

(2) (a) False. Here is a counterexample:

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}$$

You can check that $\det B = -8 = \det A$.

(b) This one is false, but tricky because it is very nearly true. Here is a counterexample:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

After switching the second and third rows we see that $\det A = -1$, but the product of the pivots is 1. It is true, however, that the determinant is equal to the product of the pivots up to a sign.

(c) False; let $B = -A$ for any $n \times n$ invertible matrix A.

(d) True. By the product rule for determinants, if the determinant of A or B is zero, so is the determinant of AB.

(3) (a) If A is $n \times n$, then $\det(-A) = (-1)^n \det A$. The matrix for T is $-I_n$. Thus $\det(T) = (-1)^n \det I_n = (-1)^n$. Therefore T is orientation preserving if n is even and orientation reversing when n is odd.

(b) $\det L^t = \det L$, so $\det(LL^t) = (\det L)^2 > 0$ since L is invertible.

(c) Let $\{\vec{v}_1, \vec{v}_2\}$ be a basis for V, and let $\hat{n}$ be the unit normal to V. These three vectors are linearly independent and thus span $\mathbb{R}^3$. We compute

$$\mathrm{ref}_V \vec{v}_1 = \vec{v}_1,\ \mathrm{ref}_v \vec{v}_2 = \vec{v}_2,\ \mathrm{ref}_V \hat{n} = \hat{n}$$

Thus in the basis $\{\vec{v}_1, \vec{v}_2, \hat{n}\}$, the matrix for ref_V is given by $\mathrm{diag}(1, 1, -1)$, where $\mathrm{diag}(a, b, c)$ is the diagonal matrix with entries a, b, c. It follows immediately that $\det \mathrm{ref}_V$ = -1 since det does not depend on the choice of basis. So, ref_V is orientation reversing no matter what V is.