110.201 Linear Algebra
4th Quiz

April 8, 2005

Problem 1 Find an orthogonal basis for the plane

$$x - y + z = 0,$$

viewed as a subspace of $\mathbb{R}^3$.

Solution So it is easy to see that the basis for the plane is

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Now let's use Gram-Schmidt algorithm to get the orthogonal basis of it.

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \textbf{then } \ |v_1| = \sqrt{v_1 \cdot v_1} = \sqrt{2}$$

So we have

$$u_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

And

$$v_2^{\perp} = v_2 - Proj_{u_1} v_2 = v_2 - (v_2 \cdot u_1) u_1 = v_2 - \frac{1}{2} v_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{bmatrix}, \textbf{and } \ |v_2^{\perp}| = \sqrt{\frac{3}{2}}$$

Therefore,

$$u_2 = \frac{v_2^{\perp}}{|v_2^{\perp}|} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{bmatrix}$$

So the orthogonal basis for the plan is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -1 \end{bmatrix}$$

Problem 2 Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the standard basis of $\mathbb{R}^3$. Consider the plane V spanned by $\vec{e}_1$ and $\vec{e}_2$.

a. For a given vector $\vec{w} = (a, b, c) \in \mathbb{R}^3$, calculate the vector $\vec{u} \in V$ that minimizes the distance between V and $\vec{w}$, i.e. find $\vec{u} \in V$ such that

$$\|\vec{u} - \vec{w}\| \leq \|\vec{v} - \vec{w}\| \quad \forall \vec{v} \in V.$$

Solution

$$\vec{u} = \mathbf{Proj}_V w = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$$

b. In the inequality above, is such a $\vec{u}$ unique for every $\vec{w}$? If so, is the assignment $\vec{w} \mapsto \vec{u}$ linear? If it is, find the matrix A of this linear transformation in the standard basis of $\mathbb{R}^3$.

Solution Yes $\vec{u}$ is unique for every $\vec{w}$. and the assignment $\vec{w} \mapsto \vec{u}$ is linear. The matrix for the linear transformation is:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 3 Find the $Q - R$ factorization of the matrix

$$A = \begin{pmatrix} 4 & 6 & 10 \\ 0 & 8 & 12 \\ 0 & 0 & 14 \end{pmatrix}.$$

Solution We can see that A is an upper triangle matrix, and all the entries in diagonal is nonzero, Therefore, the matrix is a non-degenerated matrix, so Q is I_3 and $R = A$