110.201 Linear Algebra
4th Quiz

April 8, 2005

Problem 1 Let $\vec{e}_i$, $i = 1, 2, 3, 4$ be the standard basis of $\mathbb{R}^4$. Is there an orthogonal matrix A with

$$A\vec{e}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \; A\vec{e}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \; A\vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \; A\vec{e}_4 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

or not? If so, find A, and if not, explain.

Solution By the definition of orthogonal matrix, we know

$$|Ae_4| = |e_4|$$

But

$$|Ae_4| = 2$$

while

$$|e_4| = 1$$

Therefore, the orthogonal matrix can not exist.

Problem 2 Find an orthonormal basis for the subspace $V \subset \mathbb{R}^3$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Moreover, find a normal basis for the orthogonal complement of V.

Solutions We use Gram-Schmidt to do this problem.

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

so,

$$u_1 = \frac{v_1}{|v_1|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

then,

$$v_2^{\perp} = v_2 - (v_2 \cdot u_1)u_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

Therefore,

$$u_2 = \frac{v_2^{\perp}}{|v_2^{\perp}|} = \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

The orthonormal basis of V is

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \sqrt{\frac{2}{3}} \begin{bmatrix} \frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix}$$

About the normal basis for the orthogonal complement of V, because V is 2-dimensional, so orthogonal complement must be 1-dimensional. Then let

$$v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

, so

$$v \perp u_1, v \perp u_2$$

, we have $a = -c, b = -a$ Choose

$$a = 1$$

Then we have

$$v = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

, because we need normal basis, so the answer is

$$u = \frac{v}{|v|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

Problem 3 Let $\{\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_k\}$ be a set of mutually orthogonal vectors in $\mathbb{R}^n$.

1. Show that for any $\vec{v} \in \mathbb{R}^n$, the vector

$$\vec{v} - (\text{proj}_{\vec{\alpha}_1}(\vec{v}) + \text{proj}_{\vec{\alpha}_2}(\vec{v}) + \cdots + \text{proj}_{\vec{\alpha}_k}(\vec{v}))$$

is orthogonal to each of the $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_k$.

Solution Because $\{\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_k\}$ be a set of mutually orthogonal vectors in $\mathbb{R}^n$, they are actually linear independent, then they are basis of subspace V of $\mathbb{R}^n$, with dimension k. By using Gram-Schmidt, we change the basis to orhornomal basis, because they are mutually orthogonal vectors, the orthnomal basis is just

$$\frac{\vec{\alpha}_1}{|\vec{\alpha}_1|} \cdots \frac{\vec{\alpha}_k}{|\vec{\alpha}_k|}$$

So

$$\vec{v} - (\text{proj}_{\vec{\alpha}_1}(\vec{v}) + \text{proj}_{\vec{\alpha}_2}(\vec{v}) + \cdots + \text{proj}_{\vec{\alpha}_k}(\vec{v}))$$

is a vector which is in $V^{\perp}$, orthogonal to each of the $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_k$

1. Verify this claim for $\mathbb{R}^3$ with $\vec{\alpha_1} = \vec{e_1}$, $\vec{\alpha_2} = \vec{e_2}$, and letting $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Draw a picture.

Solution

$$\vec{v} - (\text{proj}_{\vec{\alpha}_1}(\vec{v}) + \text{proj}_{\vec{\alpha}_2}(\vec{v})) = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}.$$