110.201 Linear Algebra
3rd Quiz Solutions (Thursday)

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Notation.

- P_n = space of polynomials, with real coefficients, of degree at most n.
- $\mathbb{R}^{m\times n}$ = space of m by n real matrices.

Problem 1 Determine whether the following spaces are isomorphic. In case they are isomorphic, define an isomorphism relating them. Justify your answer.
Solution Spaces are isomorphic if they have the same dimension.

1. $\mathbb{R}^2$ and $\mathbb{R}^4$. No.
2. P_5 and $\mathbb{R}^5$ No.
3. $\mathbb{R}^{2\times 3}$ and $\mathbb{R}^6$ Yes, under the natural identification.
4. P_5 and $\mathbb{R}^{2\times 3}$ Yes, under the natural identification.
5. $\mathbb{R}^{2\times k}$ and $\mathbb{C}^k$, for $k \in \mathbb{N}$. Yes, under the natural identification.

Problem 2 Let $V = \mathcal{C}^1([0,1])$ be the set of continuously differentiable functions on the closed interval $[0,1]$. V is a real linear space with respect to the operations of pointwise addition of functions and scalar multiplication.

(a) To prove that the functions $f(x) = \cos x$, $g(x) = 2x$, and $h(x) = e^x$ are linearly independent in V, consider a relation

$$c_1 \cos x + c_2 2x + c_3 e^x = 0.$$

Obviously this can only be true if $c_1 = c_2 = c_3 = 0$.

(b) Given an integer $n > 0$, $n+1$ linearly independent elements in V could be $\{2^x, 3^x, 5^x, \ldots\}$ and so on for the first $n+1$ primes.

(c) V is not isomorphic to $\mathbb{R}^m$ for any positive integer m by part (b).

Problem 3 Let $T : P_2 \to P_2$ be the linear transformation defined by

$$T(p(t)) = p''(t) + 4p'(t).$$

A typical element of P_2 is $p(t) = a + bt + ct^2$, so that $p''(t) + 4p'(t) = (2c + 4b) + (8c)t$. So the image is isomorphic to P_1, because we can write it in terms of new parameters, say $p''(t) + 4p'(t) = x + yt$, so that the rank is 2. Next, the kernal is everything that T takes to 0. So we must have $x = y = 0$, or equivalently, that $c = b = 0$. So the kernal is all polynomials of the form $p(t) = a$, which is isomorphic to P_0. So the nullity is 1, and we have rank+nullity $= 3 = \dim(P_2)$.