110.201 Quiz 3 Solutions: Friday

March 28, 2005

Problem 1

1. Yes, the map is linear. It is not an isomorphism; its kernel contains, for example, $f(x) = x$, which means the map is not one-to-one

2. The map is linear; this much is obvious. It is also clearly injective, and since an injective linear map $T : U \to V$ between linear spaces of the same finite dimension is an isomorphism, the map is an isomorphism.

3. The map is not linear. $T(kA) = k^2 T(A)$ for all real k.

Problem 2

1. There are several ways to solve this problem. One is the following: Let $\mathcal{B}_2 = \{x^4, 2x^3 - 1, 1 - x^2, 3x - 1, 2x\} \equiv \{f_1, f_2, f_3, f_4, f_5\}$. Observe that

$$
\begin{aligned}
1 &= (3/2)f_5 - f_4 && (1)\\
x &= f_5/2 && (2)\\
x^2 &= (3/2)f_5 - f_4 - f_3 && (3)\\
x^3 &= f_2/2 && (4)\\
x^4 &= f_1 && (5)
\end{aligned}
$$

Therefore $\text{span}\{f_1, f_2, \ldots, f_5\} = \text{span}\{1, x, x^2, x^3, x^4\} = P_4$. It follows immediately that the f_i are a basis of P_4, since there are only five of them and $\dim P_4 = 5$. The change-of-basis matrix from $\mathcal{B}_1$ to $\mathcal{B}_2$ can be found immediately by writing the elements of $\mathcal{B}_2$ in terms of the standard basis:

$$\begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \\ x^3 \\ x^4 \end{bmatrix}$$

S is equal to the 5×5 matrix on the right side of the above equation.

2. There was a typo on this part: T should have been written as $T(p(x)) = p'' + p' + p$. This didn't happen, so we wound up giving full credit to everyone on this part, and bonus points to anyone who noticed the mistake on their own. Here is a solution to the problem, properly stated. Immediately note that we can write $T = I + Q$, where I is the identity transformation and $Q(p) = p'' + p'$. We know what the matrix for I looks like, so we'll just work with Q. Let's compute the action of Q on the f_i, using the change of basis matrix above:

$$\begin{aligned} Q(f_1) &= 2f_2 - 12(f_3 + f_4) + 18f_5 && (6) \\ Q(f_2) &= -6(f_3 + f_4) + 21f_5 && (7) \\ Q(f_3) &= 2f_4 - 3f_5 && (8) \\ Q(f_4) &= -3f_4 + 9f_5/2 && (9) \\ Q(f_5) &= -2f_4 + 3f_5 && (10) \end{aligned}$$

Therefore the matrix for T is I_5 plus the matrix of Q in this basis, namely

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ -12 & -6 & 1 & 0 & 0 \\ -12 & -6 & 2 & -2 & -2 \\ 18 & 21 & -3 & 9/2 & 4 \end{bmatrix}$$

Problem 3 Writing out the equation defining S by performing the indicated matrix multiplication:

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \in S \iff 2x - 3y + 4z = 0$$

So, letting $x = s, y = t, z = \frac{2s-3t}{4}$, it is clear that $\dim S = 2$, and that a basis for S is given by

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -3/4 \end{bmatrix}$$

To find the dimension of S, you could also have just observed that the equation defining S is the equation of a plane in $\mathbb{R}^3$.

There are a couple of ways to do the second part. The first is to just mess around until you pull an answer out of a hat. The second is a bit sneakier and involves the observation made before: There is an isomorphism of S onto the plane P in $\mathbb{R}^3$ defined by $2x - 3y + 4z = 0$, namely

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \to \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Therefore, to find a basis of S such that A has the given representation in that basis, we can just find a basis of P such that $\vec{u} = \begin{bmatrix} 2 & 0 & -1 \end{bmatrix}$ has the representation. Then just use the above isomorphism to translate that basis back into matrix form. To get a basis like that, pick two vectors $\vec{v}_1, \vec{v}_2$ in $\mathbb{R}^3$ that are (1) not normal to P and (2) $\vec{u} = 2\vec{v}_1 + 3\vec{v}_2$. Take the projections of $\vec{v}_1$ and $\vec{v}_2$ onto P, and you get what you need. Take $\vec{v}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 & 0 & -1/3 \end{bmatrix}$. Then their projections onto P and the corresponding basis matrices are

$$\text{proj}_P \vec{v}_1 = \frac{1}{15} \begin{bmatrix} 13 \\ 3 \\ -4 \end{bmatrix} \mapsto \frac{1}{15} \begin{bmatrix} 13 & 3 \\ 0 & -4 \end{bmatrix}$$

$$\text{proj}_P \vec{v}_2 = \frac{1}{45} \begin{bmatrix} 2 \\ -3 \\ -9 \end{bmatrix} \mapsto \frac{1}{45} \begin{bmatrix} 4 & -6 \\ 0 & -7 \end{bmatrix}$$

One quickly checks that the two matrices above have the required properties.