THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences
SECOND TEST - SPRING SESSION 2005
110.201 - LINEAR ALGEBRA.

Examiner: Professor C. Consani
Duration: 50 minutes, April 27, 2005

No calculators allowed. **Total Marks = 100**

1. [**25** marks] In $\mathbb{R}^3$, find the point P on the plane described by the equation

$$x + y - z = 0$$

which is closest to $\underline{b} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Sol. Every point on the plane described by the equation $x + y - z = 0$ is a solution to

$$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

The special solutions $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are a basis for the 2-dimensional plane in $\mathbb{R}^3$.

The least squares solution $\underline{x}$ to the system

$$\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \underline{x} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

determines the point P that is closest to $\underline{b}$. Let $A = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

In particular we get

$$A^T A \underline{x} = A^T \underline{b} \qquad \text{that is}$$

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \underline{x} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \qquad \text{i.e.} \qquad \underline{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, $A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $P = (1, 0, 1)$.

2. [**25** marks] Give an orthonormal basis for the image of the linear transformation described by the matrix

$$\begin{bmatrix} 1 & 3 & 8 \\ 1 & 3 & 0 \\ 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

Sol. Use the Gram-Schmidt process:

$$\underline{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad \rightsquigarrow \quad \underline{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\underline{v}_2 = \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix} - (\underline{u}_1 \cdot \begin{bmatrix} 3 \\ 3 \\ -1 \\ -1 \end{bmatrix})\underline{u}_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ -2 \end{bmatrix} \quad \rightsquigarrow \quad \underline{u}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

$$\underline{v}_3 = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \end{bmatrix} - (\underline{u}_1 \cdot \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \end{bmatrix})\underline{u}_1 - (\underline{u}_2 \cdot \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \end{bmatrix})\underline{u}_2 = \begin{bmatrix} 4 \\ -4 \\ 0 \\ 0 \end{bmatrix} \quad \rightsquigarrow \quad \underline{u}_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}.$$

3. Consider the matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 3 & 3 & 3 \\ 4 & 4 & 4 & 4 \end{bmatrix}$

a) [**10** marks] Find $\det(A)$.

Sol. We can obtain an upper-traingular matrix via two row-exchanges. Exchange rows 1 and 4; exchange rows 2 and 3.
Two row exchanges: determinant does not change sign. The determinant of the upper-triangular matrix is: $1 \cdot 2 \cdot 3 \cdot 4 = 24$.

b) [**10** marks] Find $\det(\frac{1}{2}A)$.

Sol. $\det(\frac{1}{2}A) = (\frac{1}{2})^4 \det(A) = \frac{3}{2}$

c) [**5** marks] Is A diagonalizable? Why?

Sol. The 4 eigenvalues of A are distinct, hence A is diagonalizable.

4. Suppose the following information is known about a matrix A

$$A \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = 6 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \quad A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

a) [**10** marks] Find the eigenvalues of A.

Sol. The first two results show that 6 and 3 are eigenvalues of A. The last two results show that there are (at least) two different solutions to the system

$$A\underline{x} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}.$$

In other words, A has non-trivial nullspace, which means

$$A\underline{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \lambda_3 \underline{x}$$

where $\underline{x}$ is a non-zero vector and λ_3 must be zero. Therefore, the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = 3$, $\lambda_3 = 0$.

b) [**10** marks] Find the corresponding eigenspaces.

Sol. The first two results show that $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ are eigenvectors of A. We can find the third eigenvector that satisfies $A\underline{x} = \underline{0}$ via

$$A \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - A \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = A \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \underline{0}$$

that is $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ is the third eigenvector.

c) [**5** marks] In each of the following questions, you must give a correct reason (based on the theory of eigenvalues and eigenvectors) to get full credit

Is A a diagonalizable matrix? Is A an invertible matrix? Is A a projection matrix?

Sol. A is diagonalizable as A has distinct eigenvalues (or because the eigenvectors are linearly independent). A is not invertible, as one of the eigenvalues is zero. A is not a projection matrix, as the eigenvalues of a projection matrix are either 1 or 0.