Quiz 2 Solutions

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Problem 1 For the first part: the equation of the plane orthogonal to a given $\vec{v} \in \mathbb{R}^3$ is $\vec{x} \cdot \vec{v} = 0$. Applying this formula to our problem and writing $\vec{x} = (x, y, z)$, we get

$$x + 2y + 3z = 0.$$

For the second part, we have a formula for the reflection of $\vec{w}$ in a plane P:

$$\text{ref}_P(\vec{w}) = \vec{w} - 2\vec{w}_\perp$$

where $\vec{w}_\perp$ is the component of $\vec{w}$ perpendicular to P. But this is just

$$\vec{w} - 2(\vec{w} \cdot \hat{v})\hat{v},$$

where $\hat{v}$ is a unit vector normal to P. Applying this formula with $\hat{v} = \frac{1}{\sqrt{14}}[123]$, $\vec{w} = [456]$, we get

$$\text{ref}_P(\vec{w}) = \frac{1}{7}\begin{bmatrix} -2 \\ -25 \\ -48 \end{bmatrix}$$

Problem 2 First we find the elementary row operations that reduce A to reduced row echelon form (if possible):

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[-2(I)+(II)^*]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow[(II)+(III)^*]{} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\xrightarrow[-2(III)+(II)^*,\,(III)+(I)^*]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow[-(II)^*,\,-(III)^*]{} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since A can be reduced to the identity matrix by elementary row operations, A is invertible. Applying these same row operations to the 3×3 identity matrix gives us A^{-1}:

$$A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ -2 & 1 & 2 \\ 2 & -1 & -1 \end{bmatrix}$$

Problem 3 Let $\vec{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Note that the plane P defined by $x + z = 0$ is just the plane normal to $\vec{v}$. The problem demands a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ with kernel $\vec{v}$ and image P. Such a transformation T is given by $T(\vec{x}) = \text{proj}_P \vec{x}$. To see this, note that $\vec{v}$, being normal to P, is in the kernel of T. Moreover, $T(\mathbb{R}^3) = P$ by definition. So this is the transformation we're looking for. To find the matrix of T, write

$$\text{proj}_P \vec{x} = \vec{x} - \vec{x}_\perp = \vec{x} - (\vec{x} \cdot \vec{v})\vec{v}.$$

$$= (I - \text{proj}_{\vec{v}})\vec{x}.$$

The matrix of $\text{proj}_{\vec{v}}$ is (from HW 3)

$$B = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

Thus the matrix of T is $I - B$, or

$$\begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$