THE JOHNS HOPKINS UNIVERSITY
Faculty of Arts and Sciences

FIRST TEST - SPRING SESSION 2005

110.201 - LINEAR ALGEBRA.

Examiner: Professor C. Consani
Duration: 50 minutes, March 9, 2005

No calculators allowed. **Total Marks = 100**

1. Let A be a $(m \times n)$-matrix of rank r. Suppose $A\underline{X} = \underline{b}$ has *no solution* for some right sides $\underline{b}$ and *infinitely many solutions* for some other right sides $\underline{b}$.

(a) [**5** marks] Decide whether the nullspace of A contains only the zero vector and why.

Sol. If the nullspace of A ($N(A)$) contains only the zero vector, then $\dim N(A) = n - r = 0$, *i.e.* $n = r$ (with $n \leq m$). But then, the system could not have infinitely many solutions for some $\underline{b}$.

(b) [**5** marks] Decide whether the column space of A is all of $\mathbb{R}^m$ and why.

Sol. If the column space of A were $\mathbb{R}^m$, then the system would have always a solution, but this is in contradiction with the hypothesis that for some $\underline{b}$ the system has no solution.

(c) [**5** marks] For this matrix A find all true relations between the numbers r, m and n.

Sol. It is always true that $r \leq m$ and $r \leq n$. Moreover, under the given hypothesis, we must have $r < m$ (otherwise there would not be right sides $\underline{b}$ such that $A\underline{X} = \underline{b}$ has no solution) and $r < n$ (otherwise there would not be any $\underline{b}$ such that $A\underline{X} = \underline{b}$ has infinitely many solutions).

(d) [**5** marks] Can there be a right side $\underline{b}$ for which $A\underline{X} = \underline{b}$ has exactly one solution? Why or why not?

Sol. No, because this condition would require $n = r$.

2. (a) [**5** marks] Are the vectors $\underline{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} -8 \\ 2 \\ -2 \\ 1 \end{bmatrix}$ linearly independent? Are these vectors perpendicular to each other? Explain your answers.

Sol. Yes, the 2 vectors are linearly independent: in fact none of the two is a scalar multiple of the other one.
No, the 2 vectors are not perpendicular as $v_1 \cdot v_2 = 12 \neq 0$.

(b) [**10** marks] Do the vectors $\underline{w}_1 = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$, $\underline{w}_2 = \begin{bmatrix} 8 \\ 2 \\ 2 \\ 1 \end{bmatrix}$, $\underline{w}_3 = \begin{bmatrix} 10 \\ 1 \\ 1 \\ 6 \end{bmatrix}$, $\underline{w}_4 = \begin{bmatrix} -2 \\ -1 \\ 3 \\ 4 \end{bmatrix}$ define a basis of $\mathbb{R}^4$? Explain.

Sol. No, the vectors $\underline{w}_1$, $\underline{w}_2$, $\underline{w}_3$ and $\underline{w}_4$ are not linearly independent. For example: $c_1\underline{w}_1 + c_2\underline{w}_2 + c_3\underline{w}_3 + c_4\underline{w}_4 = \underline{0}$ for $\underline{c} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$.

(c) [**5** marks] Do the vectors $\underline{t}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\underline{t}_2 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, $\underline{t}_3 = \begin{bmatrix} -4 \\ -2 \\ 2 \\ 1 \end{bmatrix}$ define a basis of the subspace defined by (the set of solutions of) the 3-dimensional plane $x_1 + 2x_2 + 3x_3 + 6x_4 = 0$ in $\mathbb{R}^4$? Explain.

Sol. No, the vectors are not a basis for the given subspace. The vectors are linearly independent, however these vectors do not span the plane because $\underline{t}_3$ is not in the plane.

(d) [**10** marks] Find $q \in \mathbb{R}$ such that the vectors

$$\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} -1 \\ 12 \\ 10 \end{bmatrix}, \quad \begin{bmatrix} q \\ 3 \\ 1 \end{bmatrix}$$

do not span $\mathbb{R}^3$. Is this q unique? Why?

Sol. Set up the vectors as columns of a matrix and perform Gaussian elimination

$$\begin{bmatrix} 1 & 0 & -1 & q \\ 4 & 2 & 12 & 3 \\ 6 & 2 & 10 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & q \\ 0 & 2 & 16 & 3-4q \\ 0 & 2 & 16 & 1-6q \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & q \\ 0 & 2 & 16 & 3-4q \\ 0 & 0 & 0 & -2-2q \end{bmatrix}.$$

Need: $-2-2q = 0$ in the last row so that the number of non-zero pivots $= r < 3$ (r is the dimension of the column space). For $q = -1$, the four vectors span an $r = 2$ dimensional subspace of $\mathbb{R}^3$. Evidently, this value of q is unique, as it is the solution of a polynomial equation of degree 1.

3. (a) [**5** marks] Let $\underline{u} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 1 \\ 3 \\ 5 \\ 7 \end{bmatrix}$ be vectors in $\mathbb{R}^4$. Suppose we have a matrix B such that $B\underline{X} = \underline{u}$ has *no solution* and $B\underline{X} = \underline{v}$ has *no solution*, for $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$.

Is it also possible that $B\underline{X} = \underline{u} + \underline{v}$ has infinitely many solutions? If 'yes' give a matrix B that satisfies these conditions. If 'no' briefly state why the matrix B cannot exist.

Sol. Yes. Let consider the following example:

$$\begin{bmatrix} 1 & 1 \\ 5 & 5 \\ 9 & 9 \\ 13 & 13 \end{bmatrix} \underline{X} = \begin{bmatrix} 1 \\ 5 \\ 9 \\ 13 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 5 \\ 9 \\ 13 \end{bmatrix} = \underline{u} + \underline{v}.$$

This system has infinitely many solutions: $\underline{X}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\underline{X}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\underline{X}_3 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$, $\underline{X}_4 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, etc.

(b) [**5** marks] Can you find a linear transformation T_A such that Image(A) is the subspace in $\mathbb{R}^3$ described by the (set of common solutions of the) equations $x = z$, $y = 2x$ and such that $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis of Ker(A)? If 'yes' give the matrix A, if 'no' explain why this matrix cannot exist.

Sol. No. If the matrix A existed, then A would be a $m = 3$ by $n = 3$ matrix. By the Rank-Nullity Theorem, the dimension of the column space (r) plus the dimension of the nullspace ($n - r$) must equal n. However, for the assigned column space and nullspace we find that $1 + 1 \neq 3$.

(c) [**10** marks] The following information is known about a matrix B

$$B \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 3 \end{bmatrix} \quad \text{and} \quad B \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -18 \\ 9 \end{bmatrix}$$

In fact, for $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$, $B\underline{X}$ is always some multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$. What is the dimension of the nullspace of B? Give a non-zero solution to $B\underline{X} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

Sol. The column space (always some multiple of $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$) is a 1-dimensional subspace in $\mathbb{R}^2$; the dimension of the column space is $r = 1$. Note that the matrix B maps vectors in $\mathbb{R}^4$ to vectors in $\mathbb{R}^2$; then B is a $m = 2$ by $n = 4$ matrix.
Dimension of the nullspace of $B = n - r = 4 - 1 = 3$.
Moreover, note that

$$-3B \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} + B \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix} = B(-3 \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}) + B(\begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}) = B(-3 \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ -1 \\ 1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 18 \\ -9 \end{bmatrix} + \begin{bmatrix} -18 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Therefore, the vector $\underline{X}_1 = \begin{bmatrix} 0 \\ 5 \\ -8 \\ -1 \end{bmatrix}$ is a solution of the system $B\underline{X} = \underline{0}$.

(d) [**10** marks] Give a basis for the column space of $B = \begin{bmatrix} 1 & 5 & 0 & -3 \\ 2 & 10 & 1 & -4 \\ -1 & -5 & 1 & 5 \end{bmatrix}$.

Sol. By performing the Gaussian elimination we see that the pivot columns in B are: column 1 and column 3. Hence, $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ form a basis for the column space of B.

4. [**20** marks] Let V be the plane in $\mathbb{R}^3$ defined by (the set of solutions of) the equation $x - y + z = 0$. Find the matrix B of the linear transformation $T : (V, \mathcal{B}) \to (V, \mathcal{B})$, with respect to the basis $\mathcal{B} = \{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \}$ of V, which describes the orthogonal projection onto the line spanned by the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

Sol. A unit direction vector of the line is $\underline{u} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Let $\underline{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $\underline{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. The orthogonal projection onto the line with direction vector $\underline{u}$ is defined by:

$$T(\underline{x}) = (\underline{x} \cdot \underline{u})\underline{u}, \quad \underline{x} \in V.$$

In particular, we have

$$T(\underline{v}_1) = (\underline{v}_1 \cdot \underline{u})\underline{u} = \frac{1}{2}(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\underline{v}_1 + \frac{1}{2}\underline{v}_2, \quad [T(\underline{v}_1)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

Similarly,

$$T(\underline{v}_2) = (\underline{v}_2 \cdot \underline{u})\underline{u} = \frac{1}{2}(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\underline{v}_1 + \frac{1}{2}\underline{v}_2, \quad [T(\underline{v}_2)]_{\mathcal{B}} = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

It follows that $B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.