110.201 Linear Algebra
1st Quiz, Solution

February 15, 2005

Problem 1 Given the following system of equations:

$$\begin{aligned} x - 3y + z &= 1 \\ x + y + 2z &= 14 \end{aligned}$$

find all solutions using Gauss-Jordan elimination procedure. Is this an example of consistent system? Why?

Solution

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 1 & 1 & 2 & 14 \end{array}\right]$$

Row(II)-Row(I):

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 4 & 1 & 13 \end{array}\right]$$

Row(II)$\div$ 4:

$$\left[\begin{array}{ccc|c} 1 & -3 & 1 & 1 \\ 0 & 1 & \frac{1}{4} & \frac{13}{4} \end{array}\right]$$

Row (I)+ Row (II) $\times$ 3:

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{7}{4} & \frac{43}{4} \\ 0 & 1 & \frac{1}{4} & \frac{13}{4} \end{array}\right]$$

Therefore, the answer is:

$$\begin{cases} x &= \frac{43}{4} - \frac{7}{4}z \\ y &= \frac{13}{4} - \frac{1}{4}z \\ z &= \text{any real number} \end{cases}$$

Of course, consistent system and has infinite solutions.

Problem 2 Find the rank of the following matrix

$$\begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Solution

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 2 \end{bmatrix} \begin{matrix} \\ +(\text{I}) \\ \\ \text{-(I)} \end{matrix}$$

then:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \text{-(II)} \\ \\ \end{matrix}$$

then:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ \\ \times(-1) \\ \\ \end{matrix}$$

then:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} -(\text{III}) \\ -(2\times\ (\text{III})) \\ \\ \\ \end{matrix}$$

then:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here, we see, the reduced row-echelon form has 3 leading 1's, therefore, the rank is 3.

Problem 3 Show that the following linear system:

$$\begin{cases} x_1 - x_2 = b_1 \\ x_2 - x_3 = b_2 \\ x_3 - x_4 = b_3 \\ x_4 - x_5 = b_4 \\ -x_1 + x_5 = b_5 \end{cases}$$

has solution if and only if $\sum_{i=1}^{5} b_i = 0$

Proof: Use Gauss-Jordan:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & -1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & -1 & b_4 \\ -1 & 0 & 0 & 0 & 1 & b_5 \end{array}\right] +(\mathrm{I})$$

then we get:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & -1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & -1 & b_4 \\ 0 & -1 & 0 & 0 & 1 & b_5 + b_1 \end{array}\right] +(\mathrm{II})$$

then:

$$\left[\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & -1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & -1 & b_4 \\ 0 & 0 & -1 & 0 & 1 & b_5 + b_1 + b_2 \end{array}\right] +(\mathrm{III})$$

then:

$$\left[\begin{array}{ccccc:c} 1 & -1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & -1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & -1 & b_4 \\ 0 & 0 & 0 & -1 & 1 & b_5 + b_1 + b_2 + b_3 \end{array}\right] +(\text{IV})$$

then:

$$\left[\begin{array}{ccccc:c} 1 & -1 & 0 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & 0 & b_2 \\ 0 & 0 & 1 & -1 & 0 & b_3 \\ 0 & 0 & 0 & 1 & -1 & b_4 \\ 0 & 0 & 0 & 0 & 0 & b_5 + b_1 + b_2 + b_3 + b_4 \end{array}\right]$$

Therefore, the system has solution if and only if $\sum_{i=1}^{5} b_i \neq 0$