110.201 Linear Algebra
1st Quiz Solution

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Problem 1 Given the following system of equations:

$$3x + 2y - 5z = 1$$

$$4x - y + z = 0$$

$$x - z = 2$$

find all solutions using Gauss-Jordan elimination procedure. Is this an example of consistent system? Why?

Solution: We use Gauss-Jordan:

$$\left[\begin{array}{ccc|c} 3 & 2 & -5 & 1 \\ 4 & -1 & 1 & 0 \\ 1 & 0 & -1 & 2 \end{array}\right]$$

We rewrite the matrix as :

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 3 & 2 & -5 & 1 \\ 4 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} \\ -3\times\ (\mathrm{I}) \\ -4\times\ (\mathrm{I}) \end{array}$$

then:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 2 & -2 & -5 \\ 0 & -1 & 5 & -8 \end{array}\right] \div 2$$

then:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & -1 & 5 & -8 \end{array}\right] \begin{array}{l} \\ \\ +(\text{II}) \end{array}$$

then:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 4 & -\frac{21}{2} \end{array}\right] \begin{array}{l} \\ \\ \div 4 \end{array}$$

then:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 2 \\ 0 & 1 & -1 & -\frac{5}{2} \\ 0 & 0 & 1 & -\frac{21}{8} \end{array}\right] \begin{array}{l} +(\text{III}) \\ +(\text{III}) \\ \\ \end{array}$$

then we have:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{5}{8} \\ 0 & 1 & 0 & -\frac{41}{8} \\ 0 & 0 & 1 & -\frac{21}{8} \end{array}\right]$$

Therefore, we have the answer :

$$\begin{cases} x & = & -\frac{5}{8} \\ y & = & -\frac{41}{8} \\ z & = & -\frac{21}{8} \end{cases}$$

Problem 2 Find the rank of the following matrix

$$\begin{pmatrix} -1 & 3 & 8 & -2 & 1 \\ -1 & 3 & 9 & -1 & 3 \\ 1 & -3 & -9 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Solution: We find reduced row-echelon form of this matrix:

$$\begin{bmatrix} -1 & 3 & 8 & -2 & 1 \\ -1 & 3 & 9 & -1 & 3 \\ 1 & -3 & -9 & 1 & -3 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{c} \times(-1) \\ \\ +(\text{II}) \\ \div 2 \end{array}$$

then:

$$\begin{bmatrix} 1 & -3 & -8 & 2 & -1 \\ -1 & 3 & 9 & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} \\ +(\text{I}) \\ \\ \\ \end{matrix}$$

then:

$$\begin{bmatrix} 1 & -3 & -8 & 2 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} +8\times(\text{II}) \\ \\ \text{Switch with Row (IV)} \\ \text{Switch with Row (III)} \end{matrix}$$

then:

$$\begin{bmatrix} 1 & -3 & 0 & 10 & 15 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} -15\times(\text{III}) \\ -2\times(\text{III}) \\ \\ \\ \end{matrix}$$

then we have:

$$\begin{bmatrix} 1 & -3 & 0 & 10 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Therefore, we have 3 leading "1"s, that means, the rank is 3.

Problem 3 In a certain sense, the following system is not linear:

$$\begin{aligned} 2\sin\alpha - \cos\beta + 3\tan\gamma &= 3 \\ 4\sin\alpha + 2\cos\beta - 2\tan\gamma &= 10 \\ 6\sin\alpha - 3\cos\beta + \tan\gamma &= 9. \end{aligned}$$

However, there is still a way to do Gauss-Jordan elimination on it. Does a solution exist for α, β, and γ?

Solution: Let x be $\sin\alpha$, y be $\cos\beta$, and z be $\tan\gamma$

Then, we do Gauss-Jordan:

$$\left[\begin{array}{ccc:c} 2 & -1 & 3 & 3 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right] \begin{array}{l} \div 2 \\ \\ \\ \end{array}$$

then we have:

$$\left[\begin{array}{ccc:c} 1 & -1/2 & 3/2 & 3/2 \\ 4 & 2 & -2 & 10 \\ 6 & -3 & 1 & 9 \end{array}\right] \begin{array}{l} \\ -4\times \text{ (I)} \\ -6\times \text{ (I)} \end{array}$$

then:

$$\left[\begin{array}{ccc:c} 1 & -1/2 & 3/2 & 3/2 \\ 0 & 4 & -8 & 4 \\ 0 & 0 & -8 & 0 \end{array}\right] \begin{array}{c} \\ \div 4 \\ \div(-8) \end{array}$$

then:

$$\left[\begin{array}{ccc:c} 1 & -1/2 & 3/2 & 3/2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{l} +(\frac{1}{2})\times \text{ (II)} \\ \\ \\ \end{array}$$

then:

$$\left[\begin{array}{ccc:c} 1 & 0 & 1/2 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right] \begin{array}{c} -(\frac{1}{2})\times \text{ (III)} \\ +2\times \text{ (III)} \\ \\ \end{array}$$

then:

$$\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

then we get: $x = 2$, $y = 1$, $z = 0$, and because $-1 \leq \sin\alpha \leq 1$, so no solution for α, β, γ.