Midterm1 – 201, Fall 2016

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- Please do not assist another person in the completion of this exam. Please do not copy answers from another student's exam. Please do not have another student take your exam for you. Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted.
- No notes, books or calculators are allowed.
- Read the direction to each problem carefully. Show all work for full credit.
- Make sure you have **13 pages**, including the cover page (Page 1), the page of scores (Page 2), and two blank pages (Page 12 and 13).

Name: ________________ **Section:** ________________

Problem 1: (20 points):

Write down T if the following statement is true and F if the statement is false. (2 points each)

Example: F 10 vectors in $\mathbb{R}^9$ can be linearly independent.

(1) F The rank of a 7×4 matrix could be 5.

(2) T Let A be a 8×8 matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^8$. We are told that the system $A\vec{x} = \vec{b}$ has a unique solution. Then the system $A\vec{x} = \vec{c}$ has a unique solution as well.

$\Rightarrow$ A is invertible

Then $A\vec{x} = \vec{c}$ has unique solution $\vec{x} = A^{-1}\vec{c}$

(3) F The transformation $T : \mathbb{R}^2 \to \mathbb{R}$ such that $T(x_1, x_2) = x_1 - x_2 + 1$ is linear.

(4) F The diagonal matrix $\begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$ is guaranteed to be invertible if $d_1 d_2 \dots d_n \neq 0$.

(5) T Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the reflection about a plane. Then T is invertible.

$TT = I$

(6) F Consider an $n \times p$ matrix A and a $p \times m$ matrix B. $\ker(AB)$ is always contained in $\ker(B)$.

(7)
intersection $V \cap W$ is a subspace
union $V \cup W$ is not

(7) T Consider two subspaces V and W of $\mathbb{R}^n$. The ~~union~~ intersection $V \cap W$ is necessarily a subspace of $\mathbb{R}^n$.

(6). $\ker B = \{\vec{x} : B\vec{x} = \vec{0}\}$

$\ker AB = \{\vec{x} : AB\vec{x} = \vec{0}\}$

Since $B\vec{x} = \vec{0}$ implies $AB\vec{x} = \vec{0}$,

then $\ker B$ is contained in $\ker(AB)$.

The opposite is not necessarily true.

Try $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(8) T Consider some perpendicular vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ in $\mathbb{R}^n$. These vectors are necessarily linearly independent.

(9) F Let V be the subspace of $\mathbb{R}^{10}$ defined by the homogeneous equation $x_1 - 2x_2 + 3x_3 - 4x_4 + 5x_5 - 6x_6 + 7x_7 - 8x_8 + 9x_9 - 10x_{10} = 0$. The dimension of V is 1.

$\dim V = 9$

(10) T Let V be the space of all infinite sequences of real numbers. The subset of sequences $(x_0, x_1, \ldots)$ that converge to zero (i.e., $\lim_{n\to\infty} x_n = 0$) is a subspace of V.

Problem 2 (28 points):

Consider the linear system:

$$\begin{aligned} x \quad &+ \ (a^2-4)z = a+3 \\ x+y+ &\ (a^2-4)z = a+5 \\ x \quad &+ (2a^2-8)z = 2a+5 \end{aligned} .$$

(1) Write down the augumented matrix. Perform elementary row operations to obtain the Row Echelon Form (REF). Recall that pivots are not necessarily 1 in REF. (5 points)

$$\left[\begin{array}{ccc|c} 1 & 0 & a^2-4 & a+3 \\ 1 & 1 & a^2-4 & a+5 \\ 1 & 0 & 2a^2-8 & 2a+5 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & a^2-4 & a+3 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & a^2-4 & a+2 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & a^2-4 & a+2 \end{array}\right]$$

(2) For which value(s) of a is this system inconsistent? (7 points)

When $a=2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \end{array}\right]$$

No solution.

(3) For which value(s) of a does this system have a unique solution? Write down the solution. (8 points)

If $a^2-4 \neq 0 \Rightarrow a \neq \pm 2$, rank $A = 3$,

the system has a unique solution.

$X_1 = 1 \qquad X_2 = 2 \qquad X_3 = \dfrac{a+2}{a^2-4}$

(4) For which value(s) of a does this system have infinitely many solutions? Find all the solutions. (8 points)

When $a = -2$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2. \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Infinite solution $X_1 = \ \ X_2 = 2 \quad X_3 = t$.

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ t \end{bmatrix}$$

Problem 3 (12 points):

Find the inverse of the following linear transformation:

$$T\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+3z \\ x+2y+6z \\ x+6z \end{bmatrix}. = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 6 \\ 1 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$T\vec{v} = A\vec{v} = W$$

$$T^{-1}W = A^{-1}W$$

$$\begin{array}{c} \\ ②-① \\ ③-① \end{array}\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 1 & 2 & 6 & 0 & 1 & 0 \\ 1 & 0 & 6 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & -1 & 1 & 0 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

①-③
②-③

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & 0 & 1 & -1 \\ 0 & 0 & 3 & -1 & 0 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{matrix} 2 & 0 & -1 \\ 0 & 1/2 & -1/2 \\ -1/3 & 0 & 1/3 \end{matrix}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

Problem 4 (24 points):

$$A = \begin{bmatrix} 1 & -2 & 0 & 2 & 0 & 0 \\ 1 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{bmatrix}.$$

(1) Perform elementary row operations to obtain the RREF(A). (4 points)

pivots

$$\begin{bmatrix} 1 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 3 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(2) Find a basis for Image(A). What is the dimension of Image(A)? (10 points)

Basis of Image (A) is 1st, 3rd, 6th column.

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

(3) Find a basis for Ker(A). What is the dimension of Ker(A)? (10 points)

$$\ker A = \left\{ \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} : A\vec{x} = \vec{0} \right\}$$

$$x_6 = 0.$$

$$x_3 = -2x_4 - 3x_5$$

$$x_1 = 2x_2 - 2x_4$$

Basis of kerA

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2x_2 - 2x_4 \\ x_2 \\ -2x_4 - 3x_5 \\ x_4 \\ x_5 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Problem 5 (16 points):

(1) Consider the space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Are the following functions linearly independent? Why?

$$\{x, \sin x, \cos x, x - \sin x + 2\cos x\}.$$

(6 points)

No. , since $x - \sin x + 2\cos x$ is a linear combination of x, $\sin x$, $\cos x$.

(2) Two 2×3 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \end{bmatrix}$$

are perpendicular if

$$a_{11}b_{11} + a_{12}b_{12} + a_{13}b_{13} + a_{21}b_{21} + a_{22}b_{22} + a_{23}b_{23} = 0.$$

Find a basis for the space of all 2×3 matrices that are perpendicular to the following matrices:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & 0 \end{bmatrix}.$$

(10 points)

$A \perp \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$ $\quad a_{11} + 2a_{22} = 0 \quad a_{11} = -2a_{22}$

$A \perp \begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ $\quad 3a_{12} - a_{23} = 0 \quad a_{23} = 3a_{12}$

$A \perp \begin{bmatrix} 0 & 0 & -4 \\ 1 & 0 & 0 \end{bmatrix}$ $\quad -4a_{13} + a_{21} = 0 \quad a_{21} = 4a_{13}$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} -2a_{22} & a_{12} & a_{13} \\ 4a_{13} & a_{22} & 3a_{12} \end{bmatrix}$$

$$= a_{22}\begin{bmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} + a_{12}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} + a_{13}\begin{bmatrix} 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}$$

Basis