Midterm1 – 201, Fall 2016

Instructor: Wenjing Liao

- Please keep your own work covered up as much as possible during the exam so that others will not be tempted or distracted.
- No notes, books or calculators are allowed.
- Read each problem carefully. Show all work for full credit.
- Make sure you have **10 pages**, including the cover page (Page 1) and the score page (Page 2).

Name: ________________ **Section:** ________________

I would not assist anybody else in the completion of this exam. I would not copy answers from others. I would not have another student take the exam for me.

Signature: ________________

(1)	
(2)	
(3)	
(4)	
(5)	
(6)	
Total	

Problem 1: (20 points):

Write down T if the following statement is true and F if the statement is false. (2 points each)

Example: F 10 vectors in $\mathbb{R}^9$ can be linearly independent.

(1) ______ The transformation $T(A) = A^T$ from $\mathbb{R}^{5\times 5}$ to $\mathbb{R}^{5\times 5}$ is an isomorphism.

(2) ______ Consider the linear transformation:

$$T(\vec{x}) = \det \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots & \vec{v}_{n-1} & \vec{x} \end{bmatrix}$$

from $\mathbb{R}^n$ to $\mathbb{R}$, where $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{n-1}$ are linearly independent vectors in $\mathbb{R}^n$. The nullity of T is equal to n.

(3) ______ Determinant of the following matrix is necessarily 0.

$$\begin{bmatrix} a_1b_1 & a_1b_2 & a_1b_3 & a_1b_4 \\ a_2b_1 & a_2b_2 & a_2b_3 & a_2b_4 \\ a_3b_1 & a_3b_2 & a_3b_3 & a_3b_4 \\ a_4b_1 & a_4b_2 & a_4b_3 & a_4b_4 \end{bmatrix}$$

(4) ______ The dimension of $W^{\perp}$ is equal to 2, where

$$W = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ -3 \\ -4 \end{bmatrix} \right).$$

(5) ______ The following matrix is an orthogonal matrix.

$$\begin{bmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{bmatrix}$$

(6) ______ If the $n \times n$ matrices A and B are orthogonal matrices, $A + B$ must be orthogonal as well.

(7) ______ Consider an orthonormal basis $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$ in $\mathbb{R}^n$. The least-squares solution of the system $A\vec{x} = \vec{u}_n$ where $A = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_{n-1}]$ is $\vec{x} = \vec{0}$.

(8) ______ $\langle f, g \rangle = f(1)g(1) + f(2)g(2)$ is an inner product in P_1.

(9) ______ Let A be an $n \times n$ matrix. If λ is an eigenvalue of A, then $\lambda^2 + \lambda + 1$ is an eigenvalue of the matrix $A^2 + A + I_n$.

(10) ______ There is an orthogonal transformation T from $\mathbb{R}^3$ to $\mathbb{R}^3$ such that

$$T \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}.$$

Problem 2 (12 points):

Let
$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$
(1) Find the eigenvalues of A. (5 points)

(2) Find the eigenvectors of A. (7 points)

Problem 3 (16 points):

Let

$$A = \begin{bmatrix} -1 & 1 \\ 1 & 3 \\ 1 & 3 \\ -1 & 1 \end{bmatrix}.$$

(1) Find the QR factorization of A. (10 points)

(2) Let $V = \text{image}(A)$. Compute $\text{Proj}_V \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$. (6 points)

Problem 4 (20 points, 5 points each):

Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and $B = \begin{pmatrix} a & d & g \\ b & e & h \\ 1 & 2 & 3 \end{pmatrix}$ such that $\det A = 2$ and $\det B = 3$.

(1) Compute $\det \begin{pmatrix} 0 & 0 & 1 & -2 \\ a & b & c & 1 \\ d & e & f & 2 \\ g & h & i & 3 \end{pmatrix}$.

(2) Suppose $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ is the solution of $A\vec{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Evaluate x_3.

(3) Let $C = (ABA^{-1})^{2016}$. Compute $\det C$.

(4) Compute $\det E$, where E is the reduced row echelon form of B.

Problem 5 (20 points):

Consider the linear transformation $T : P_3 \to P_3$ such that
$T(a + bt + ct^2 + dt^3) = a + dt + (b + c)t^2 + (a + d)t^3$.

(1) Given the basis $\mathfrak{B} = \{1, t, t^2, t^3\}$ in P_3. Find the $\mathfrak{B}$-matrix of T with respect to the basis $\mathfrak{B}$. (8 points)

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(2) Find a basis of the kernel of T. (6 points)

(3) Find a basis of the image of T. (6 points)

Problem 6 (12 points):

Let $\vec{e}$ be a unit vector in $\mathbb{R}^3$. Consider the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T(\vec{x}) = 2(\vec{e} \cdot \vec{x})\vec{e} - \vec{x}$.

(1) Given a geometric interpretation of T. In other words, express T in terms of reflections, projections, or rotations. (4 points)

(2) Is T an isomorphism? Why? (4 points)

(3) Is T an orthogonal transformation? Why? (4 points)