# Final Review Outline

December 9, 2016

**Test format:** There will be 13 problems and you choose to do 12 problems. All problems are computational. There are no True/False questions. Notice that some problems in midterms, after being modified, will appear in final.

**The following kind of problems are likely to appear:**

1. Solve a linear system.
2. Given a subspace in $\mathbb{R}^n$, find a basis for the subspace and determine its dimension. Best practice problems are HW problems in Section 3.3.
3. Given a linear transformation, find a basis for the kernel and image of the linear transformation.
4. Given a linear transformation and a basis $\mathfrak{B}$, write down the $\mathfrak{B}$-matrix. Find a basis for the image and kernel of the linear transformation. Best practice problems are HW problems in Section 4.3 and Problem 5 in Midterm 2.
5. Given a pairwise operation on a linear space, determine whether it is an inner product. In an inner product space, compute the orthogonal projection of an element onto a subspace using Theorem 5.5.3. Best practice problems are HW problems in Section 5.5.
6. Compute the determinant of a square matrix.
7. Orthogonal diagonalization of a symmetric matrix. Best practice problems are HW problems in Section 8.1.
8. Given a quadratic form, write down the matrix $A$ representing the quadratic form, and determine its definiteness. Best practice problems are HW problems in Section 8.2.
9. Compute Singular Value Decomposition of a matrix. Best practice problems are HW problems in Section 8.3.
10. Solve a least squares problem. Best practice problems are HW problems in Section 5.4.

**The following kind of problems are not on final:**

1. Change of basis.
2. Fourier coefficients.
3. Complex eigenvalues.

**Content:** Final covers Chapter 1, Chapter 2, Chapter 3, Chapter 4, Chapter 5, Chapter 6, Section 7.1, 7.2, 7.3, and Chapter 8.

**Chapter** 1

- Section 1.1 and 1.2: Solving a linear system. Given a linear system, write down the augumented matrix, perform elementary row operations on the augumented matrix to solve the linear system.
- Section 1.2 and 1.3: The Reduced Row Echelon Form (RREF) of a matrix; Perform elementary row operations to obtain the RREF of a matrix; Decide whether a linear system has a unique solution, infinitely many solutions, or no solution from the RREF of the augumented matrix.
- Section 1.3: Matrix operations: Matrix addition, scalar multiplication, vector dot product, matrix vector product.
- Section 1.3: Properties of matrix operations.
- Section 1.3: The rank of a matrix.

**Chapter** 2

- Section 2.1: Definition of linear transformations, Determine whether a transformation is linear or not.
- Section 2.1: Given a linear transformation, write down the matrix representing the linear transformation.
- Section 2.2: Linear transformation in geometry: scaling, orthogonal projection, reflection, rotation.
- Section 2.3: Matrix product: compute matrix multiplication, write matrix product in terms of rows of the first matrix or columns of the second matrix (Theorem 2.3.2). Properties of matrix product. Notice that matrix multiplication is non-commmutative.
- Lecture notes: Definition of one-to-one and onto transformation. Given a linear transformation, determine whether it is one-to-one or onto. A linear transformation $T : R^n \to R^m$ such that $T\mathbf{x} = A\mathbf{x}$ is:

  $$\text{one-to-one} \Leftrightarrow \text{Kernel}(T) = \{\mathbf{0}_{R^n}\} \Leftrightarrow \text{rank}(A) = n.$$

  A linear transformation $T : R^n \to R^m$ such that $T\mathbf{x} = A\mathbf{x}$ is:

  $$\text{onto} \Leftrightarrow \text{Image}(T) = R^m \Leftrightarrow \text{rank}(A) = m.$$

- Section 2.4: definition of invertible linear transformations and invertible matrices. Theorem 2.4.3. Find the inverse of a matrix. Properties of inverse.

**Chapter** 3

- Section 3.1: Definition of kernel. If $T : R^n \to R^m$ is a linear transformation, then Kernel($T$) is a subspace of $R^n$. Given a linear transformation, find its kernel.
- Section 3.1: Definition of image. If $T : R^n \to R^m$ is a linear transformation, then Image($T$) is a subspace of $R^m$. Given a linear transformation, find its image.
- Section 3.2: Definition of subspace. Given a subset of $R^n$, determine whether it is a subspace.

- Section 3.2: Definition of linear dependence and linear independence. Given a set of vectors, determine whether they are linearly independent.
- Section 3.2: Definition of bases. Given a set of vectors, determine whether they form a basis of $R^n$. The dimension of a subspace is the number of vectors in the basis.
- Section 3.3: Given a linear transformation, find a basis of the kernel, and the image of the linear transformation, respectively. Find the dimension of the kernel, and the image respectively.
- Section 3.3: If $T : R^n \to R^m$ is a linear transformation, then

$$\dim \text{kernel}(T) + \dim \text{Image}(T) = n.$$

- Section 3.4: Definition of coordinates. Given a set of basis, find the coordinates of a vector; Given a linear transformation and a set of basis, find the matrix of the transformation with respect to the given basis.

**Chapter 4**

- Section 4.1: Definition of a vector space. You are not required to check whether a set is a vector space.
- Section 4.1: Definition of subspace in a vector space. Given a subset in a vector space, determine whether it is a subspace.
- Section 4.1: Linear independence/dependence, linear combinations, span, basis, dimension and coordinates.
- Section 4.2: Linear transformation: given a transformation between two vector spaces, determine whether it is a linear transformation.
- Section 4.2: Image, kernel, rank and nullity of a linear transformation. Given a linear transformation, find bases of the image and the kernel, determine the rank and nullity. Keep in mind that

$$\text{rank} + \text{nullity} = \dim(\text{input space}).$$

- Section 4.2: Isomorphism: an invertible linear transformation is called an isomorphism. Given a transformation between two vector spaces, determine whether it is an isomorphism.
- Section 4.3: The $\mathfrak{B}$-matrix of a linear transformation (Definition 4.3.1). Given a linear transformation and a basis $\mathfrak{B}$, write down the $\mathfrak{B}$-matrix. Use this matrix to find bases of the image and kernel of the linear transformation. Also use this matrix to determine whether a linear transformation is invertible. See Example 4 in Section 4.3.
- Section 4.3: Change of basis is not on the exam.

**Chapter 5**

- Section 5.1: Definition of orthonormal vectors. $n$ orthonormal vectors in $\mathbb{R}^n$ form a basis of $\mathbb{R}^n$.
- Section 5.1: Orthogonal projection to a subspace (Theorem 5.1.4). Given a subspace, compute the orthogonal projection onto it (Theorem 5.1.5).
- Section 5.1: Orthogonal complement: Definition 5.1.7 and Theorem 5.1.8.
- Section 5.2: Given $m$ linearly independent vectors $v_1, \ldots, v_m$, run Gram-Schmidt to obtain orthonormal vectors $u_1, \ldots, u_m$; or given an $n \times m$ matrix with linearly independent columns, compute the QR factorization.

- Section 5.3: Orthogonal transformations and orthogonal matrices. Orthogonal transformations preserve the length of vectors, and the dot product between two vectors. Theorem 5.3.3 and Theorem 5.3.4. Determine whether a linear transformation is an orthogonal transformation, or whether a matrix is an orthogonal matrix. Theorem 5.3.7 and Summary 5.3.8. Properties of the transpose (Theorem 5.3.9).
- Section 5.4: Theorem 5.4.1 and the applications. Theorem 5.4.2 and Theorem 5.4.3. Given an inconsistent linear system, find the least-squares solution (Theorem 5.4.5). Applications of least squares include finding certain functions that best fit given data points.
- Section 5.5: Definition of inner product and inner product spaces. Given a pairwise operation, determine whether it is an inner product. In an inner product space, compute the inner product, norm and orthogonal projection (Example 4,5,6,7).
- Notice that we focus on $C[-\pi,\pi]$ with the inner product $\langle f,g\rangle = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)g(t)dt$. Section 5.5: Find the Fourier series of a given function.

**Chapter 6**

- Definition of determinant. Notice that in the lecture we define determinant through Laplace expansion in Theorem 6.2.10 (see my lecture notes as well). The definition of determinant in Section 6.1 is not requried.
- Section 6.2: The relation of elementary row operations and determinants (Theorem 6.2.3). Use Gauss-Jordan elimination to compute the determinant (Algorithm 6.2.5).
- Section 6.2: Properties of determinant: Theorem 6.2.1, Theorem 6.2.2, Theorem 6.2.6, Theorem 6.2.7 and Theorem 6.2.8.
- Section 6.3: Geometric interpretation of determinants (Theorem 6.3.6). Use determinant to compute the volume of a parallelepiped. Theo determinant as expansion factor (Theorem 6.3.7).
- Section 6.3: Cramer's rule.

**Chapter 7**

- Section 7.1: Eigenvectors, eigenvalues and eigenbases (Definition 7.1.2). Diagonalization (Theorem 7.1.3).Summary 7.1.5.
- Section 7.2: Given a matrix, compute the characteristic polynomial, eigenvalues and eigenvectors of the matrix.
- Section 7.3: Algebraic multiplicity and geometric multiplicity of eigenvalues. Diagonalization: given a square matrix, determine whether it is diagonalizable (Theorem 7.3.3 and Theorem 7.3.4). Given a square matrix, diagonalize it (Theorem 7.3.7).

**Chapter 8**

- Section 8.1: Symmetric matrices and the spectral theorem (Theorem 8.1.1). Properties of symmetric matrices: Theorem 8.1.2 and Theorem 8.1.3.
- Section 8.1: Orthogonal diagonalization of a symmetric matrix (Theorem 8.1.4).
- Section 8.2: Quadratic forms (Definition 8.2.1). Given a quadratic form, write down the matrix $A$ representing the quadratic form.
- Section 8.2: Diagonalization of a quadratic form (Theorem 8.2.2).

- Section 8.2: Definiteness of a quadratic form (Definition 8.2.3). Given a quadratic form or a symmetric matrix, determine its definiteness.
- Section 8.3: Singular values (Definition 8.3.1, Theorem 8.3.3 and Theorem 8.3.4). Singular value decomposition (Theorem 8.3.5): given a matrix, compute its singular value decomposition.