**Math 201**
**Fall 2014**
**Midterm 1**
**10/08/14**
**Lecturer: Jesse Gell-Redman**
**Time Limit: 50 minutes**

**Name (Print):** ____________________

Teaching Assistant ____________________

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This exam contains 10 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Show your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Follow the instructions closely**. For example, if you are asked to justify your answers, then do so in a brief and coherent way.
- **Points will be taken off for incorrect statements, even if correct ones are present.** Be careful about what you include in your answers. If they contain both the correct answers and incorrect or nonsense statements, points will be taken off.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

**Good luck!!** Do not write in the table to the right.

| Problem | Points | Score |
|---|---|---|
| 1 | 20 | |
| 2 | 20 | |
| 3 | 20 | |
| 4 | 0 | |
| 5 | 20 | |
| Total: | 80 | |

1. (20 points) **Solving linear systems:** show your work.

   (a) (10 points) Find all solutions to the linear system

$$\begin{array}{rcrcrcr} x & + & 4y & + & 4z & = & 5 \\ & & y & + & z & = & 1 \\ -x & - & 2y & - & 2z & = & -3 \end{array}$$

**Solution**

We perform Gauss-Jordan elimination:

$$\left(\begin{array}{ccc|c} 1 & 4 & 4 & 5 \\ 0 & 1 & 1 & 1 \\ -1 & -2 & -2 & -3 \end{array}\right) \xrightarrow{(III):+(I)} \left(\begin{array}{ccc|c} 1 & 4 & 4 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{array}\right) \xrightarrow[(III):-2(II)]{(I):-4(II)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From this RREF matrix, we read that $x = 1$ and $y + z = 1$, indicating that solutions are of the form $(1, t, 1 - t)$ for $t \in \mathbb{R}$.

(b) (10 points) Consider the linear system

$$\begin{array}{rcrcrcr} x & + & y & - & z & = & -2 \\ 3x & - & 5y & + & 13z & = & 18 \\ x & - & 2y & + & 5z & = & k \end{array}$$

where $k$ is a real number. For what values of $k$ does this system admit solutions, i.e. when is it consistent? (You don't need to solve it.)

**Solution**

We perform Gauss-Jordan elimination:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 3 & -5 & 13 & 18 \\ 1 & -2 & 5 & k \end{array}\right) \xrightarrow[(III):-(I)]{(II):-3(I)} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & -8 & 16 & 24 \\ 0 & -3 & 6 & k+2 \end{array}\right)$$

$$\xrightarrow{(II):\div(-8)} \left(\begin{array}{ccc|c} 1 & 1 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & 6 & k+2 \end{array}\right) \xrightarrow[(III):+3(II)]{(I):-(II)} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & k-7 \end{array}\right)$$

From this RREF matrix, we read that if $k \neq 7$ then the system is inconsistent. If $k = 7$, there are infinitely many solutions of the form $(1-t, -3+2t, t)$ for $t \in \mathbb{R}$

2. (20 points) Number of solutions

$$\text{Let } A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 0 & -1 & -1 \end{pmatrix}$$

(a) (10 points) Given any $\vec{b} \in \mathbb{R}^3$, can you say whether or not there are solutions to $A\vec{x} = \vec{b}$? If so, how many? Justify your answers.
**Solution:** We start by row reducing $A$.

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{(II):-(I)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -1 \end{pmatrix} \xrightarrow{(III):+(II)} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

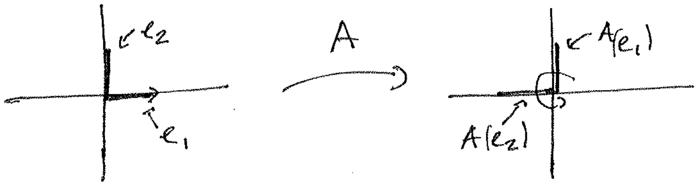
$$\xrightarrow[(II):-2(I)]{(I):-2(II)} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{(I):+(III)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Because $A$ row reduces to the identity, it is invertible. Thus $A\vec{x} = \vec{b}$ has a unique solution.

(b) (10 points) Let $B$ be a $4 \times 4$ matrix, and let $\vec{b}$ and $\vec{c}$ be two vectors in $\mathbb{R}^4$. Assume that $B\vec{x} = \vec{b}$ is *inconsistent*. What can you say about the number of solutions to the equation $B\vec{x} = \vec{c}$.
**Solution:** There are either zero or infinitely many solutions. Because the system is inconsistent, we get a row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}$ when row reducing the augmented matrix $(B|b)$ (note that this is not necessarily the last row).In particular when row reducing $B$ we get at least one row of zeros. Hence row reducing $(B|c)$ gives at least one row of the form $\begin{pmatrix} 0 & 0 & 0 & | & 1 \end{pmatrix}$ or of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$. If the row reduced echelon form of $(B|c)$ contains a row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}$, then $B\vec{x} = \vec{c}$ is inconsistent. If it contains no row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & | & 1 \end{pmatrix}$, and it does contain a row of the form $\begin{pmatrix} 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$, then it has infinitely many solutions. In either case, it cannot have a unique solution.
Alternatively, we could instead have shown $B\vec{x} = \vec{c}$ does not have a unique solution as follows. Since $B\vec{x} = \vec{b}$ has no solution, $B$ is not invertible. Hence $\ker B \neq \{0\}$. This implies the solution to $B\vec{x} = \vec{c}$ is not unique, because if $\vec{x}$ is a solution to this equation and $\vec{a} \in \ker B$, then $\vec{x} + \vec{a}$ is also a solution.
We have shown that the solution cannot be unique. To see that zero solutions or infinitely many solutions are possibilities, it suffices to consider $B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, and either $\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ or $\vec{c} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$.

3. (20 points) **Linear transformations of the plane**

(a) (5 points) Write down the matrix $A$ which gives rotation counter-clockwise by $90°$.

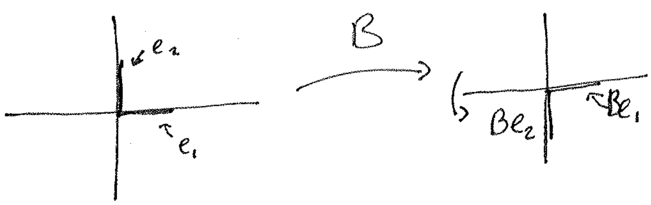



$\Rightarrow A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

or $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = 90°$ $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

people reflected about the x-axis

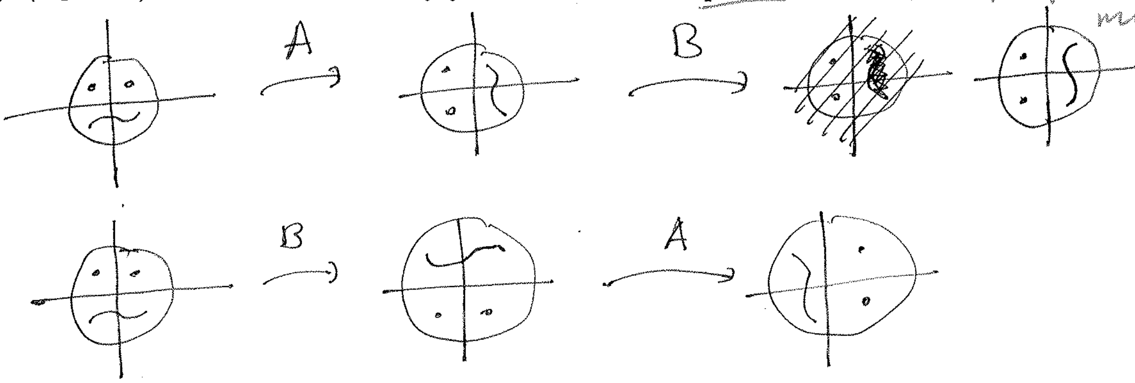
(b) (5 points) Write down the matrix $B$ that gives reflection $\boxed{\text{over}}$ the x-axis.




$\Rightarrow B = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

often pictures were missing
some people did matrix mult. only

(c) (5 points) Is $AB = BA$? Justify your answer with a picture.




(d) (5 points) Find $A^{-1}$. Explain geometrically the linear tranformation $T\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that has $T(\vec{v}) = A^{-1}\vec{v}$.

$\left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \end{array}\right] \xrightarrow{-R_2} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{array}\right]$

T is rotation clockwize by $90°$

4. (20 points) **Image, kernel, and dimension:** Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 8 \\ -1 & -1 & -5 \end{pmatrix}$$

(a) (8 points) Find a basis for image($A$). What is dim(image($A$))?

**Solutionsx** First we row reduce

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 8 \\ -1 & -1 & -5 \end{pmatrix} \xrightarrow[(III):+(I)]{(II):-2(I)} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -2 \end{pmatrix} \xrightarrow{(III):+(II)} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

A basis for image($A$) can be obtained by choosing the columns of $A$ corresponding to the pivot columns of RREF($A$). So, image($A$) has basis

$$\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

The common mistakes for this problem was arithmetic errors while doing the row reduction, and choosing the vectors $e_1$ and $e_2$ from RREF($A$) instead of the corresponding columns in $A$

(b) (7 points) Find a basis for ker($A$).

**Solution:** We need the row reduction of $A$, luckily we already figured it out. We know that $\ker(A) = \ker(\mathrm{RREF}(A))$, so all we need is to find the kernel of RREF($A$)

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+3z \\ y+2z \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Choosing $z = 1$ we obtain

$$\begin{pmatrix} -3 \\ -2 \\ 1 \end{pmatrix} \in \ker(A)$$

We know that ker($A$) is one dimensional so, this vector forms a basis.

(c) (5 points) For *any* $3 \times 3$ matrix $B$, is it possible to have

$$\dim(\mathrm{image}(B)) = \dim(\ker(B))?$$

If no, why not? If yes, give an example.

**Solution:** Suppose yes, there is a $B$ that satisfies the above conditions then:

$$\dim(\mathrm{image}(B)) + \dim(\ker(B)) = 3$$

$$2\dim(\mathrm{image}(B)) = 3$$

Therefore for such a $B$ to exist 3 would have to be divisible by 2 or the dimension would be a non-integer. Therefore as both of these are clearly false such a $B$ cannot exist.

Common mistakes: People omitted to mention the small amount of algebra. Often people said "3 would have to be even" which is valid but they didn't have the algebra on the page to back it up. Also, the fact that we're talking about the linear dimension of vector spaces and that has to be intergral is essential.

5. (20 points) Consider the vectors

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{in } \mathbb{R}^3.$$

Prove that

$$\operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) = \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

By arguing along the lines discussed in class. Namely, recall that given two sets $X$ and $Y$ in $\mathbb{R}^n$, to prove that $X = Y$, we argued separately that $X \subset Y$ ("$X$ is contained in $Y$") and that $Y \subset X$ ("$Y$ is contained in $X$").

**Proof:**

Let

$$X := \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right) \text{ and let } Y := \operatorname{span}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right).$$

To keep notation simple, let

$$\vec{v}_1 := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \vec{v}_2 := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \vec{v}_3 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Recall that a vector $\vec{v}$ lies in $X$ if and only if it is a linear combination of $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$, meaning that there exist real numbers $c_1, c_2$, and $c_3$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3. \tag{1}$$

Similarly, $\vec{v}$ lies in $Y$ if and only if it is a linear combination of $\vec{v}_1$, and $\vec{v}_2$, meaning that there exist real numbers $d_1$, and $d_2$ such that

$$\vec{v} = d_1\vec{v}_1 + d_2\vec{v}_2 \tag{2}$$

*Step 1, $Y \subset X$:* This step is easier than the next. Assume that $\vec{v} \in Y$. We wish to show that also $\vec{v} \in X$. (Remember that "$\in$" means "is contained in".) As we just stated, $\vec{v} \in Y$ means exactly that there are numbers $d_1$ and $d_2$ so that the second equation above, equation (2), holds. To show that $\vec{v}$ is also in $X$, we need only know that there are number $c_1, c_2$, and $c_3$ so that equation (1) holds, but indeed one can just take $c_1 = d_1$, $c_2 = d_2$, and $c_3 = 0$.

*Step 2, $X \subset Y$:* Let $\vec{v} \in X$. We wish to show that $\vec{v} \in Y$ also. Again, saying that $\vec{v} \in X$ is the same as saying that there are $c_1, c_2$, and $c_3$ so that (1) holds, and to show that $\vec{v} \in Y$ we need to find $d_1, d_2$ so that equation (2) holds.

Note that

$$\vec{v}_3 = \vec{v}_1 - \vec{v}_2 \text{ since } \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and so we can rewrite (1) as follows

$$\begin{aligned}\vec{v} &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \\ &= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - \vec{v}_2) \\ &= (c_1 + c_3)\vec{v}_1 + (c_2 - c_3)\vec{v}_2.\end{aligned} \tag{3}$$

But this final expression shows exactly that $\vec{v} \in Y$, since taking $d_1 = c_1 + c_3$ and $d_2 = c_2 - c_3$ shows that equation (2) holds (i.e. $\vec{v}$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$.)

**On the grading:** One did not have to write nearly as much as I did here to get full credit on this problem. Partial credit was given for many things, especially for either stating clearly or otherwise showing that you understood the definition of span, for pointing out or showing that the vectors $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ are linearly independent, and for making an attempt to argue the two inclusions separately. If you just pointed out that the vectors were dependent and said "so you can remove one from the span," you probably did not get many points as the problem asks you to prove that statement (and indeed tells you along which lines to argue.)