LINEAR ALGEBRA
First Midterm Exam
SOLUTIONS

JOHNS HOPKINS UNIVERSITY
SPRING 2013

You have **50 MINUTES.**
No calculators, books or notes allowed.

Academic Honesty Certificate. I agree to complete this exam without unauthorized assistance from any person, materials or device.

Signature: ______________________ Date: __________

Name: ______________________ Section №: __________
(or TA's name)

Question	*Score*
1	
2	
3	
4	
5 (bonus)	

(1) (a) [15 points] Find all solutions to the system of equations:

$$\begin{cases} x + y + 6z = 8 \\ 2x + 3y + 16z = 21 \end{cases}$$

using Gaussian elimination. Is the system consistent? Why?

Answer. Perform Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 1 & 6 & 8 \\ 2 & 3 & 16 & 21 \end{array}\right] \begin{array}{l} \\ -2(\mathrm{I}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 1 & 6 & 8 \\ 0 & 1 & 4 & 5 \end{array}\right] \begin{array}{l} -(\mathrm{II}) \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & 4 & 5 \end{array}\right]$$

The system is consistent since there are no rows of the form $0 \cdots 0|1$ in the reduced row eschelon form. In fact the general solution is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - 2z \\ 5 - 4z \\ z \end{bmatrix}$$

□

(b) [10 points] Does the system of equations:

$$\begin{cases} 2x + 12z = 14 \\ 2y + 16z = 18 \\ x + 2y + 22z = 25 \end{cases}$$

have a unique solution? Justify your answer.

Answer. Perform Gaussian elimination on the augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 0 & 12 & 14 \\ 0 & 2 & 16 & 18 \\ 1 & 2 & 22 & 25 \end{array}\right] \begin{array}{l} \div 2 \\ \div 2 \\ -(\mathrm{II}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \\ 1 & 0 & 6 & 7 \end{array}\right] \begin{array}{l} \\ \\ -(\mathrm{I}) \end{array} \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 6 & 7 \\ 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since the system is consistent and since the third column is a free column, the system has infinitely many solutions. So it does *not* have a unique solution. □

(2) [25 points] Let:

$$\mathrm{A} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 5 \\ 1 & 1 & 1 & 15 \end{bmatrix}$$

Can an equation:

$$\mathrm{A}\vec{x} = \vec{b}$$

have infinitely many solutions while another equation:

$$\mathrm{A}\vec{x} = \vec{c}$$

has none whatsoever? If no, explain why not. If yes, find vectors $\vec{b}$ and $\vec{c}$ in $\mathbf{R}^4$ for which this is true.

Answer. Perform Gaussian elimination on the augmented matrix with indeterminates $b_1, \ldots, b_4$ in the rightmost column:

$$\left[\begin{array}{cccc:c} 1 & -1 & 0 & 0 & b_1 \\ 0 & 1 & -1 & 0 & b_2 \\ 0 & 0 & 1 & 5 & b_3 \\ 1 & 1 & 1 & 15 & b_4 \end{array}\right]\begin{array}{l} +(\mathrm{II}) \\ +(\mathrm{III}) \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{cccc:c} 1 & 0 & -1 & 0 & b_1+b_2 \\ 0 & 1 & 0 & 5 & b_2+b_3 \\ 0 & 0 & 1 & 5 & b_3 \\ 1 & 1 & 1 & 15 & b_4 \end{array}\right]\begin{array}{l} +(\mathrm{III}) \\ \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc:c} 1 & 0 & 0 & 5 & b_1+b_2+b_3 \\ 0 & 1 & 0 & 5 & b_2+b_3 \\ 0 & 0 & 1 & 5 & b_3 \\ 1 & 1 & 1 & 15 & b_4 \end{array}\right]\begin{array}{l} \\ \\ \\ -(\mathrm{I})-(\mathrm{II})-(\mathrm{III}) \end{array}$$

$$\rightsquigarrow \left[\begin{array}{cccc:c} 1 & 0 & 0 & 5 & b_1+b_2+b_3 \\ 0 & 1 & 0 & 5 & b_2+b_3 \\ 0 & 0 & 1 & 5 & b_3 \\ 0 & 0 & 0 & 0 & b_4-b_1-2b_2-3b_3 \end{array}\right]$$

The system is consistent if & only if $b_4 - b_1 - 2b_2 - 3b_3 = 0$. So the answer to the question is 'yes'. We can construct the desired vectors $\vec{b}$ and $\vec{c}$ by setting $b_1 = b_2 = b_3 = 1$ and choosing b_4 so as to satisfy or not satisfy the condition, e.g.:

$$\vec{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 6 \end{bmatrix} \qquad \vec{c} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \square$$

(3) (a) [5 points] Compute the matrix products BA and AB where:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Answer.

$$BA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad AB = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \square$$

(b) [10 points] Does the matrix A above have an inverse? If yes, compute it. If no, why not?

Answer. We can determine whether A has an inverse and compute that inverse (if it exists) simultaneously by performing Gaussian elimination on the matrix $[A|I_3]$:

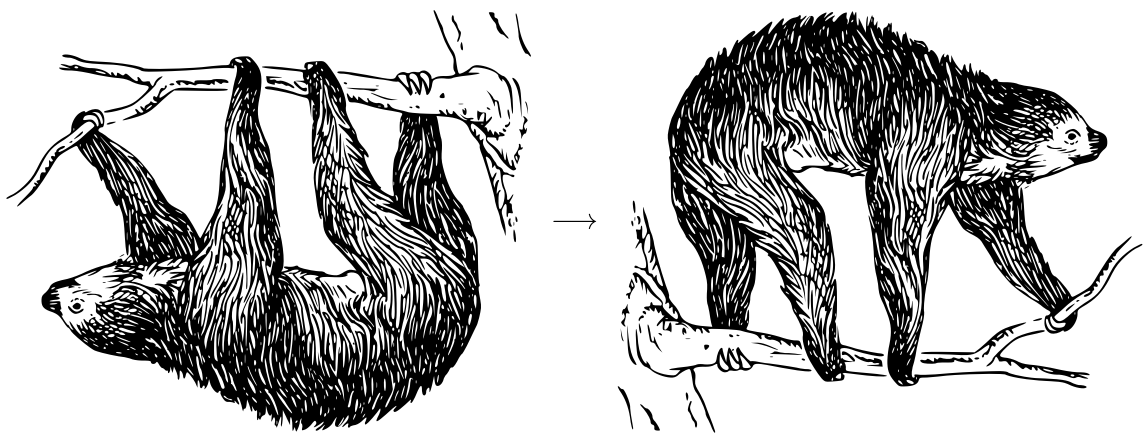
$$\left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -(\mathrm{III}) \\ \\ \\ \end{array} \rightsquigarrow \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \text{move to the bottom} \\ \\ \\ \end{array}$$

$$\rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

Since we obtained I_3 on the left, the matrix A *is* invertible, and its inverse is the bit on right:

$$A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \qquad \square$$

(3) (c) [10 points] Write down the matrix for the following linear transformation. (The origin is at the center of each drawing.) Explain how you reached your answer.



[This must be how sloths look to one another!]

Answer. Thinking about the branch and the sloth's legs we see easily that:

$$\vec{e}_1 \mapsto -\vec{e}_1 \qquad\qquad \vec{e}_2 \mapsto -\vec{e}_2$$

So the corresponding matrix is:

$$\begin{bmatrix} | & | \\ T(\vec{e}_1) & T(\vec{e}_2) \\ | & | \end{bmatrix} = \begin{bmatrix} | & | \\ -\vec{e}_1 & -\vec{e}_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$ □

(4) (a) [5 points] What does it mean to say that vectors $\vec{v}_1, \ldots, \vec{v}_n$ are *linearly independent?*

Definition. The vectors $\vec{v}_1, \ldots, \vec{v}_n$ are linearly independent if the only solution to the equation:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_n\vec{v}_n = 0$$

is $x_1 = x_2 = \cdots = x_n = 0$. □

(b) [5 points] Are these vectors linearly independent? Justify your answer using determinants.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad\qquad \vec{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Answer. The vectors are linearly independent if and only if the equation:

$$\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \vec{x} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \vec{x} = \vec{0}$$

has a unique solution $\vec{x}$. Since we are dealing with a 2×2 matrix, this is equivalent to asking whether its determinant is nonzero. So simply compute:

$$\det \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = 1 \cdot 3 - 2 \cdot 2 = 3 - 4 = -1 \neq 0$$

So the vectors are indeed linearly independent. □

(4) (c) [15 points] What is the dimension of the space spanned by the following vectors? Explain your approach and show your work.

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \qquad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \\ 3 \end{bmatrix}$$

Answer. The space spanned by vectors $\vec{v}_1, \ldots, \vec{v}_4$ is the same thing as the image of the matrix:

$$\mathrm{A} = \begin{bmatrix} | & | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \\ | & | & | & | \end{bmatrix}$$

Since the dimension of the image of a matrix equals its rank, all we need to do is compute the rank of A, equivalently the number of leading columns in rref(A). We do this by performing some row reduction:

$$\mathrm{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 2 & -1 \\ 2 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 \\ 2 & 1 & 1 & 3 \end{bmatrix} \begin{matrix} \\ -(\mathrm{III}) \\ -2(\mathrm{I}) \\ -2(\mathrm{I}) \\ -(\mathrm{III}) \end{matrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Swapping the 2nd and 3rd rows makes it clear that A has rank 3. $\square$

(5) [20 bonus points] Find a basis for the image of the linear transformation:

$$\mathrm{A} = \begin{bmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{bmatrix}$$

for any real numbers a and b.

[*Hint: The special cases* $(a,b)=(0,0)$ *and* $(a,b)=(0,1)$ *immediately show that the number of basis vectors will depend on the values a and b take, so carry out as much row reduction as possible without dividing by possibly vanishing numbers and break into cases at the last step.*]

Answer. Perform some row reduction but avoid dividing by any possibly vanishing numbers:

$$\begin{bmatrix} a & a & b & a \\ a & a & b & 0 \\ a & b & a & b \\ a & b & a & 0 \end{bmatrix} \begin{matrix} -(\mathrm{I}) \\ \\ -(\mathrm{IV}) \\ -(\mathrm{II}) \end{matrix} \rightsquigarrow \begin{bmatrix} 0 & 0 & 0 & a \\ a & a & b & 0 \\ 0 & 0 & 0 & b \\ 0 & b-a & a-b & 0 \end{bmatrix} \overset{\text{swap rows}}{\rightsquigarrow} \begin{bmatrix} a & a & b & 0 \\ 0 & b-a & a-b & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & b \end{bmatrix}$$

Now break into cases.

$\boxed{\text{If } a \neq 0 \text{ and } b \neq a}$ then the 1st, 2nd & 4th columns are leading columns so a basis for the image would be the 1st, 2nd & 4th columns of A:

$$\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}$$

$\boxed{\text{If } a \neq 0 \text{ and } a = b}$ then only the 1st & 4th columns are leading columns so a basis for the image would be:

$$\begin{bmatrix} a \\ a \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}$$

$\boxed{\text{If } a = 0 \text{ and } b \neq a}$ then the 2nd & 4th *but also the 3rd* column are leading columns so a basis for the image would be:

$$\begin{bmatrix} a \\ a \\ b \\ b \end{bmatrix}, \begin{bmatrix} b \\ b \\ a \\ a \end{bmatrix}, \begin{bmatrix} a \\ 0 \\ b \\ 0 \end{bmatrix}$$

$\boxed{\text{If } a = 0 \text{ and } b = a}$ then none of the columns are leading columns so the image equals $\{\vec{0}\}$ whose basis is *empty*. □