CHALLENGE PROBLEM SET: CHAPTER 6, SECTIONS 1 AND 2, COURSE WEEK 11

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. For what values of the constant k are the following matrices invertible:

$$\textbf{(a)} \begin{bmatrix} 4 & 0 & 0 \\ 3 & k & 0 \\ 2 & 1 & 0 \end{bmatrix}, \qquad \textbf{(b)} \begin{bmatrix} 0 & 1 & k \\ 3 & 2k & 5 \\ 9 & 7 & 5 \end{bmatrix}, \qquad \textbf{(c)} \begin{bmatrix} 1 & 1 & 1 \\ 1 & k & k \\ k & k & k \end{bmatrix}.$$

Question 2. For the following matrices $\mathbf{A}$, use the determinant to determine for which values of the constant λ the matrix $\mathbf{A} - \lambda \mathbf{I}_3$ fails to be invertible:

$$\textbf{(a)} \quad \mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 5 & 3 & 0 \\ 7 & 6 & 4 \end{bmatrix}, \qquad \textbf{(b)} \quad \mathbf{A} = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 4 & 2 \\ 0 & 2 & 7 \end{bmatrix}.$$

Question 3. Calculate the determinants of the following 5×5 matrices:

$$\textbf{(a)} \begin{bmatrix} 5 & 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 & 0 \\ 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix}, \qquad \textbf{(b)} \begin{bmatrix} 0 & 0 & 1 & 0 & 2 \\ 5 & 4 & 3 & 2 & 1 \\ 1 & 3 & 5 & 0 & 7 \\ 2 & 0 & 4 & 0 & 6 \\ 0 & 0 & 3 & 0 & 4 \end{bmatrix}, \qquad \textbf{(c)} \begin{bmatrix} 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & 0 & 2 & 2 \\ 0 & 9 & 7 & 9 & 3 \\ 0 & 0 & 0 & 0 & 5 \\ 3 & 4 & 5 & 8 & 5 \end{bmatrix}.$$

Question 4. If $\mathbf{A}$ is an $n \times n$ matrix, and k is an arbitrary constant, establish the relationship between $\det(\mathbf{A})$ and $\det(k\mathbf{A})$.

Question 5. Let $\mathbf{M}_n$ be the $n \times n$ matrix with all ones along "the other diagonal", and zeros everywhere else. for example,

$$\mathbf{M}_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

(a) Find $\det(\mathbf{M}_n)$, for $n = 2, 3, 4, 5, 6, 7$.

(b) Find a formula for $\det(\mathbf{M}_n)$, in terms of n.

Question 6. Do the following:

(a) Find the determinant of the following via Gaussian elimination:

$$\textbf{(i)}\quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}, \qquad \textbf{(ii)}\quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 3 & 3 \\ 1 & 1 & 1 & 4 & 4 \\ 1 & 1 & 1 & 1 & 5 \end{bmatrix}.$$

(b) If a 4×4 matrix $\mathbf{A}$ has columns $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$, and $\det(\mathbf{A}) = 8$, then find the determinant of the following:

$$\textbf{(i)}\ \det\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \\ -9\mathbf{v}_4 \end{bmatrix}, \quad \textbf{(ii)}\ \det\begin{bmatrix} \mathbf{v}_2 \\ \mathbf{v}_3 \\ \mathbf{v}_1 \\ \mathbf{v}_4 \end{bmatrix}, \quad \textbf{(iii)}\ \det\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 + 9\mathbf{v}_4 \\ \mathbf{v}_4 \\ \mathbf{v}_3 \end{bmatrix}, \quad \textbf{(iv)}\ \det\begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \\ \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 \end{bmatrix}.$$

Question 7. Find the determinants of the linear transformations:

(a) $T : P_2 \to P_2$, $T(f(t)) = f(-t)$.

(b) $T : P_3 \to P_3$, $T(f(t)) = f(-t)$.

(c) $T : \mathbb{C} \to \mathbb{C}$, $T(z) = (1 + 2i)z$.

(d) $T : V \to V$, $T(\mathbf{M}) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\mathbf{M}$, where V is the space of upper triangular 2×2 matrices.

(e) $T : W \to W$, $T(f) = 3f - 2f' + f''$, where $V = \operatorname{span}\{\cos t, \sin t\}$.

Question 8. Consider the two distinct real numbers a and b. Do the following for the function

$$f(t) = \det\begin{bmatrix} 1 & 1 & 1 \\ a & b & t \\ a^2 & b^2 & t^2 \end{bmatrix}:$$

(a) Show that $f(t)$ is a quadratic function. What is the coefficient of t^2?

(b) Explain why $f(a) = f(b) = 0$. Conclude that $f(t) = k(t - a)(t - b)$ for some constant k. Find k.

(c) For which values of t is the matrix invertible?

Question 9. Completely answer the following questions via explanation:

(a) Let $\mathbf{A}_{n\times n}$ be invertible such that both $\mathbf{A}$ and $\mathbf{A}^{-1}$ have integer entries. What are the possible values of $\det(\mathbf{A})$?

(b) For $\mathbf{A}$ invertible, what are the possible values of $\det(\mathbf{A}^T\mathbf{A})$?

(c) If $\mathbf{A}$ is orthogonal, what are the possible values of $\det(\mathbf{A})$?

(d) Let $\mathbf{A}$ be a skew-symmetric $n \times n$ matrix, where n is odd. Can $\mathbf{A}$ possibly be invertible? Why or why not?

Question 10. Give an example of a 3×3 matrix $\mathbf{A}$ with all nonzero entries where $\det(\mathbf{A}) = 13$.