CHALLENGE PROBLEM SET: CHAPTER 5, SECTIONS 3 AND 5, COURSE WEEK 10

110.201 LINEAR ALGEBRA
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Question 1. Find all orthogonal 2×2 matrices.

Question 2. Do the following:

(a) Find an orthogonal transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that $T \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) Is there an orthogonal transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$T \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} \quad \text{and} \quad T \begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}.$$

Question 3. Do the following:

(a) Consider the line $L \subset \mathbb{R}^n$ spanned by the vector $\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$. Consider the matrix $\mathbf{A}$ of the orthogonal projection onto L. Describe the ijth entry of $\mathbf{A}$ in terms of the components u_i of $\mathbf{u}$.

(b) Find the matrix $\mathbf{A}$ of the orthogonal projection onto the line in $\mathbb{R}^n$ spanned by the vector

$$\left.\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right\} \text{all } n \text{ components are 1.}$$

Question 4. Do the following:

(a) Find a basis of the space of all symmetric 4×4 matrices, and determine the dimension of the space.

(b) Find a basis of the space of all skew-symmetric 4×4 matrices, and determine the dimension of the space.

(c) Find the image and kernel of the linear transformation $L(\mathbf{A}) = \frac{1}{2}\left(\mathbf{A} - \mathbf{A}^T\right)$, from $\mathbb{R}^{4\times4}$ to $\mathbb{R}^{4\times4}$. *Hint:* This is part **(c)** for a reason.

Question 5. In $\mathbb{R}^4$, consider the subspace W spanned by the vectors $\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$. Find the matrix $\mathbf{P}_W$ of the orthogonal projection onto W.

Question 6. Consider an $n \times m$ matrix $\mathbf{P}$ and an $m \times n$ matrix $\mathbf{Q}$. Show that

$$\text{trace}(\mathbf{PQ}) = \text{trace}(\mathbf{QP}).$$

Question 7. Do the following:

(a) Find an *orthonormal* basis of the space P_1 with inner product

$$\langle f, g\rangle = \int_0^1 f(t)g(t)\, dt.$$

(b) Show that $\langle f, g\rangle = \frac{1}{2}\left(f(0)g(0) + f(1)g(1)\right)$ is an inner product on P_1.

(c) Find an *orthonormal* basis of the space P_1 with the inner product

$$\langle f, g\rangle = \frac{1}{2}\left(f(0)g(0) + f(1)g(1)\right).$$

Question 8. For which 2×2 matrices $\mathbf{A}$ is

$$\langle \mathbf{v}, \mathbf{w}\rangle = \mathbf{v}^T\mathbf{A}\mathbf{w}$$

an inner product on $\mathbb{R}^2$? *Hint:* Be prepared to complete the square.

Question 9. Questions to argue over:

(a) If T is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ such that $T(\mathbf{e}_1), \ldots, T(\mathbf{e}_n)$ are all unit vectors, then T must be an orthogonal transformation.

(b) If $\mathbf{u}$ is a unit vector in $\mathbb{R}^n$, and $L = \text{span}(\mathbf{U})$, then $\text{proj}_L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\mathbf{x}$ for all vectors $\mathbf{x} \in \mathbb{R}^n$.

(c) If a matrix $\mathbf{A}$ is orthogonal, then $\mathbf{A}^T$ must be orthogonal as well.

(d) Every invertible matrix $\mathbf{A}$ can be expressed as a product of an orthogonal matrix and an upper triangular one.

(e) The entries of an orthogonal matrix are all less than or equal to 1.

(f) If $\mathbf{A}$ is an invertible matrix such that $\mathbf{A}^{-1} = \mathbf{A}$, then $\mathbf{A}$ must be orthogonal.

(g) Any square matrix can be written as the sum of a symmetric matrix and a skew-symmetric matrix.

(h) There exists a basis of $\mathbb{R}^{2\times 2}$ that consists of orthogonal matrices.