CHALLENGE PROBLEM SET: CHAPTER 4, SECTIONS 2 AND 3, COURSE WEEK 8

110.201 LINEAR ALGEBRA
PROFESSOR RICHARD BROWN

Question 1. Consider the following three problems as the same type of problem. In each instance, divide into four groups working in parallel on the board (or work out the problems as one group), and determine whether each function described below is a linear transformation. For those that are, determine whether they are isomorphisms. Compare and contrast your solutions and reasoning:

(a) As transformations from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$:

$$\textbf{(i)} \quad T(\mathbf{M}) = \mathbf{M}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\mathbf{M}.$$

$$\textbf{(ii)} \quad T(\mathbf{M}) = \mathbf{M}\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\mathbf{M}.$$

$$\textbf{(iii)} \quad T(\mathbf{M}) = \mathbf{M}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{M}.$$

$$\textbf{(iv)} \quad T(\mathbf{M}) = \mathbf{M}\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{M}$$

(b) As transformations from $\mathbb{C}$ to $\mathbb{C}$:

$$\textbf{(i)} \quad T(x+iy) = -y, \qquad \textbf{(ii)} \quad T(x+iy) = x^2+y^2,$$
$$\textbf{(iii)} \quad T(x+iy) = x - iy, \quad \textbf{(iv)} \quad T(x+iy) = y + ix.$$

(c) As transformations from P_2 to P_2:

$$\textbf{(i)} \quad T(f(t)) = f(-t), \qquad \textbf{(ii)} \quad T(f(t)) = f(2t),$$
$$\textbf{(iii)} \quad T(f(t)) = f(2t) - f(t), \quad \textbf{(iv)} \quad T(f(t)) = f''(t)f(t).$$

Question 2. For which values of $k \in \mathbb{R}$ is the linear transformation

$$T(\mathbf{M}) = \begin{bmatrix} 2 & 3 \\ 0 & 4 \end{bmatrix}\mathbf{M} - \mathbf{M}\begin{bmatrix} 3 & 0 \\ 0 & k \end{bmatrix}$$

and isomorphism.

Question 3. Using the basis $\mathcal{B} = \{1, x, x^2\}$ for P_2, find a matrix B, relative to $\mathcal{B}$, so that the linear transformations can be written in coordinates. Then use this to determine whether the transformations are isomorphisms or not. When not an isomorphism, find bases for the image and kernel of the transformation:

(a) $T(f(t)) = f'(t)$,
(b) $T(f) = f' - 3f$,
(c) $T(f(t)) = f(-t) - f''(t)$,
(d) $T(f(t)) = f(2t-1)$,
(e) $T(f(t)) = f(2t-1)$, with new basis $\mathcal{B} = \{1, t-1, (t-1)^2\}$.
(f) $T(f(t)) = f(1) + f'(1)(t-1)$, with new basis $\mathcal{B} = \{1, t-1, (t-1)^2\}$.

Question 4. Let V be the linear space of all functions spanned by the two functions $\sin x$ and $\cos x$. For the following transformations, find the matrix of the transformation relative to the basis $\{\sin x, \cos x\}$. Determine whether the transformations are isomorphisms:

(a) $T(f) = f'' + af' + bf$, where a and b are arbitrary real numbers.

(b) $T(f(t)) = f\left(t - \frac{\pi}{2}\right)$.

(c) $T(f(t)) = f(t - \theta)$, where θ is an arbitrary real number. *Hint:* Use the addition identities for sine and cosine.

Question 5. Let V be the linear space of all functions of the form

$$f(t) = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t).$$

Consider the linear transformation T from V to V given by

$$T(f) = f'' + f.$$

(a) Find the matrix for T with respect to the basis given by $\cos(t)$, $\sin(t)$, $t\cos(t)$, and $t\sin(t)$ of V.

(b) Find all solutions $f(t)$ in V of the differential equation

$$T(f) = f'' + f = \cos(t).$$

Graph your solution(s). (Note here that the differential equation $f'' + f = \cos(t)$ describes a forced, undamped oscillator. In this example, we observe a physical phenomenon called *resonance*. You will solve this again using techniques of calculus in 110.302 Differential Equations.

Question 6. Questions to argue over:

(a) The space P_1 is isomorphic to $\mathbb{C}$.

(b) The function $T(\mathbf{M}) = \det(\mathbf{M})$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}$ is a linear transformation.

(c) The linear transformation $T(\mathbf{M}) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \mathbf{M}$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$ has rank 1.

(d) If the image of a linear transformation T from P to P (the space of all polynomials) is all of P, then T must be an isomorphism.

(e) If f_1, f_2, and f_3 form a basis for a linear space V, then f_1, $f_1 + f_2$, and $f_1 + f_2 + f_3$ must form a basis for V as well.

(f) The linear transformation $T(f(t)) = f(4t - 3)$ from P to P is an isomorphism.

(g) Every 2-dimensional subspace of $\mathbb{R}^{2\times 2}$ contains at least one invertible matrix.

(h) The transformation $D(f) = f'$ from $C^\infty(\mathbb{R})$ to $C^\infty(\mathbb{R})$ is an isomorphism.